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TEACHERS' MANUAL

ANSWERS to Textbook Tests and Achievement Tests

Second Course in ALGEBRA

with TRIGONOMETRY

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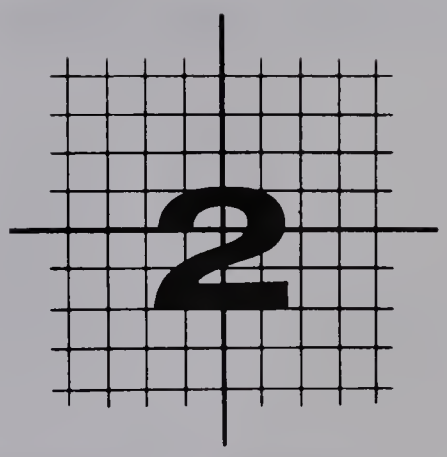

Second Course in ALGEBRA

WEEKS • ADKINS

with TRIGONOMETRY

GINN AND COMPANY BOSTON • NEW YORK • CHICAGO • ATLANTA • DALLAS • PALO ALTO • TORONTO

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TEACHERS' MANUAL

GENERAL REMARKS

A second course in algebra can be quite an exciting experience. In such a course the structure of the subject becomes evident and familiar. Increasing awareness of this structure in a variety of situations gives the uninitiated their first feeling for the great generality of the subject. The shift from open sentences in one variable to sentences that define a set of ordered pairs of numbers widens enormously the range of problems to be dealt with. Another link in the general understanding of numbers is forged when the one-to-one correspondence between ordered pairs of numbers and points of the plane is postulated. The trigonometric functions take their place in the general structure. The method of mathematical induction intrigues the curious. Sequences and series, including the binomial series, constitute the last basic step before the calculus.

The great amount of practice that ensues within this general framework leads to an exhilarating increase in mathematical power and a satisfying realization of understanding. The great job of the teacher is to keep the peaks unobscured by the clouds of detail and to get out of the way when the eager student starts moving!

The position of the course in the general curricular sequence may vary from school to school. It may follow immediately a first course in algebra; a year of geometry may precede it; or it may be spread over two years, being taught concurrently during that time with the geometry. (See pages 16-28 for a discussion of a two-year course combining second-course algebra, geometry, and trigonometry.) Whatever the curricular sequence may be, the mastery of this book and of A Course in Geometry, Plane and Solid will prepare the student well for a beginning course in calculus in either grade twelve or grade thirteen.

Ordinary college-capable classes will find enough in the book to allow some choice of topics in the course of a year. A very good class can complete the entire book. Plan C, top of next page, shows a usable division of time for completing the book in 180 school days. Plan A shows a distribution of time for a course which does not attempt to finish the trigonometry. Plan

B shows a course which completes the trigonometry but touches certain other topics more lightly than does Plan A.

Chapter	Plan A (Days per chapter)	Plan B (Days per chapter)	Plan C (Days per chapter)
1	3	3	3
2	5	5	3
3	10	9	7
4	10	9	7
5	8	7	6
6	7	6	5
7	8	7	6
8	20	15	15
9	10	10	8
10	12	8	11
11	5	5	0
12	15	15	13
13	17	17	20
14	15	10	15
15	10	10	9
16	0	12	10
17	15	10	15
18	10	10	15
19	0	12	12
	180	180	180

The teacher is strongly advised to read the book through in a couple of sittings. This will give a "feeling" for the book and a general basis for keeping the various details in some sensible perspective. As a further effort toward a general point of view we here list the chapter headings and then "talk" our way from chapter to chapter.

Chapter

1. Rational Numbers
2. Equations and Inequalities
3. Systems of Linear Equations; Word Problems
4. Factored Forms
5. Fractions
6. Quadratic Equations with Rational Roots
7. Formulas; General Conclusions
8. Irrational Numbers

9. Functions, Graphs, Variation
10. Exponents and Logarithms
11. Measurement in Right Triangles
12. Elements of Coordinate Geometry
13. The Quadratic Function
14. Equations of the Second Degree and Their Graphs
15. The Sine and Cosine Functions; Oblique Triangles
16. Trigonometric Equations and Identities
17. Polynomials
18. Sequences and Series
19. Trigonometric Addition Formulas

The first seven chapters are a review, at a more sophisticated level, of the content of a first course in algebra. Chapter 1 stresses the nature of a number system in general and of the system of rational numbers in particular. It proves a few theorems in order to give the student an idea of the "flavor" of an algebraic proof. The Sidelight exhibits a non-numerical system consisting of four elements which, in classes of superior ability, may be used to introduce the concept of a group.

To talk about the elements of our number system we must obviously use sentences. The kinds of sentences we shall use are equations and inequalities. In Chapter 2 we recall what was previously studied about equations in one variable, emphasizing the ways of solving by obtaining a sequence of equivalent equations and the ways of solving inequalities. When equalities are described by words, we can often simplify matters by using two or more variables instead of one. This leads us into systems of linear equations. Very soon, in the solution of equations, we meet expressions whose factored form is desirable for comfortable manipulation. Fractions can be handled after factoring is available, and Chapter 5 therefore deals with fractions, and in a more formal manner than does the First Course in Algebra. Polynomials of the first degree lead naturally to polynomials of the second degree, and Chapter 6 makes this transition to quadratic equations with rational roots. We are now able to deal with a considerable variety of formulas, with an emphasis on the process of deducing new information from the general statement presented by the formula. The concepts of ratio and proportion extend the variety of problems that can be handled.

Throughout the first seven chapters the universe of discourse

is the set of rational numbers. General study of functions of real numbers is desired, however; and the number system is therefore expanded by the introduction of irrational numbers in Chapter 8. As usual, the main steps in such an expansion of the number system are to define the new numbers, and then to define the fundamental operations with them, so that they fit smoothly into the system of rational numbers. And finally, enough practice must be given with the new numbers so that the student can use them without trouble.

With irrational numbers added to the number system, the way is open for completing the number line, developing from it the coordinate plane, and then for postulating the one-to-one correspondence between ordered pairs of real numbers and points of the plane. If a set of ordered pairs is defined by an equation $y = f(x)$, then the typical element is (x, y). The set of all x is the domain and the set of all y is the range. If for every x-value in the set of ordered pairs there is a unique corresponding y-value, then the set of all (x, y) is a function. The focus of attention now shifts from sets of numbers to sets of ordered pairs of numbers, and the rest of the book is basically a study of functions: linear, quadratic, trigonometric, and higher-degree polynomial, with technical details brought in when the need for them arises.

Comment on the Nature of the Exercises

A study of the nature of the exercises will help the teacher to make an efficient use of the book. The exercises have been carefully graded, so that reliance can be placed on the "A" and "B" designations. There are so many of them that it should be unnecessary for the teacher to construct more. There are many spots where the types of problems in an exercise have been arranged in pairs so that the assignment of the odd or of the even numbers in a set will give maximum variety. It is also easy to select problems from the book for additional tests.

Finally, there are many places in the exercises where topics are forecast prior to their formal presentation in the text. This helps the students to discover an idea before it is formally presented to them. Following are some examples of this "forecasting" technique:

pp. 153-154: Exs. such as 22-25, and Ex. 23 in the next group lead up to the solution of quadratics with irrational roots, taught on pp. 157 ff.

p. 185: Exs. 20-21 anticipate the midpoint formula, taught on p. 266.

pp. 185-186: Ex. 2(a) and Exs. 2, 5, and 7 in set A, pp. 187-188 lead up to the idea of slope. So do Exs. 8 and 9, where we have first concurrent lines and then parallel lines.

pp. 224-225: Exs. 13 ff. lead up to the following section on logarithms.

pp. 272-273: Exs. 18, 19, and 20 in set A-1, and Exs. 15, 16, and 17 in set A-2 lead up to the criteria for parallel and perpendicular lines.

pp. 320-321: The B exercises lead up to the teaching in the next sections.

pp. 343-345: Exs. 47-50 in set A-1, and Exs. 33-36 in set A-2 lead up to the section on sum and product of roots, p. 348.

p. 373: Ex. 5 in set A-1 and Ex. 4 in set A-2 lead up to the general investigation of "Line Tangent to a Curve" on pp. 378 ff.

p. 407: Exs. 19, 20, and 21 lead up to the discussion on pp. 408 ff.

p. 410: Exs. 1-4 lead up to "Special Angles" on page 412.

p. 411: Ex. 19 in set A foreshadows one of the Pythagorean relationships on p. 435.

p. 418: Ex. 14 leads to the next section on the Law of Sines.

pp. 463-464: Exs. 16 and 17 in set A-1 and Exs. 17-19 in set A-2 lead to the next section on "Approximate Solution of Cubic Equations."

p. 539: Exs. 17, 18, and 19 foreshadow identities, introduced on pp. 546-547.

Digression on the Role of Vocabulary in the Learning of Mathematics

The subject of mathematics fits properly into general education when we can find a vocabulary to convey its broad ideas to students. This presumably refers, at this level of maturity, to a vocabulary that will tell the students about the relations of number, the relations of quantity, and the relations of space. Such a vocabulary will certainly need to include many words that summarize in themselves a great body of ideas and generalizations. The word "function" is an example.

Although the teacher may prefer to introduce the word "function" before the quadratic function is studied, there is little need for the word until such relations as $y = \pm\sqrt{x}$ are met. When this time comes, there will have been enough experience with sets of ordered pairs of numbers to make the usefulness of the word "function" apparent to the student.

Vocabulary is useful for putting new ideas or topics into context. For instance, in introducing the study of polynomial functions in general, and the cubic in particular, the teacher gives sense, coherence, and motivation to the study by reminding the class of their study of the quadratic function. In that study, points of similarity to the more general problems of the cubic, as well as points of contrast, enable the student to put the investigation into context. Thus, a natural motivation for the study of the polynomial is provided.

In the development of new ideas it is essential that the "timing" for the use of words be carefully thought out. Experience with a concept or idea must precede the verbalization of the idea or concept. Just how much experience is best cannot be stated in general. Certainly the timing was off somewhere for the boy who said that:

> "A locus is when the point moves on to the curve and stays there until the equation is satisfied."

And then there is the very real question of whether words should ever be used to complete the learning of such a relationship as the distributive axiom. Certainly the name "distributive axiom" is useful, somewhere along the line, but perhaps the symbolism $a(b + c) = ab + ac$ is the best way to summarize the great number of such examples with which one deals.

The quantity of experiences with which it is desirable to precede verbalization varies with the nature of the idea or word. For example: "reciprocal" is used almost immediately, but "function" may be long postponed. There is a theorem in most geometry books which says: "If, in a right triangle, the altitude is drawn to the hypotenuse, then the altitude is the mean proportional between the projections of the legs on the hypotenuse, and either leg is the mean proportional between the hypotenuse and the projection of that leg on the hypotenuse." Here is a case where verbalization might well never occur. Instead, the more basic notion of ratios of corresponding sides of similar triangles might better be used to obtain any individual result.

In learning generalizations, contrast and variation are important. For example, there would seem to be little point in talking about a "perfect square trinomial" if the pupil has not met trinomials that are not perfect squares. A definition of rectangle has little motivation if there has been no experience

with quadrilaterals that are not rectangles. People are now pretty well agreed that the student's understanding of equations is helped by contrasting equations with inequalities. "Truth" in mathematics is clarified by contrasting truth with "falsity." Age level and the purpose of the course are factors to be considered in employing contrast and variation.

In mathematics every new idea has some kind of connection with old ideas. The flow from old ideas to new ones is therefore moderately smooth. The language we use ought to make this flow plain, easy to follow, and satisfyingly coherent. A simple and obvious example is the property of the relation of equality known as symmetry. This property takes one smoothly from multiplication to factoring. It provides a unifying thread with past experience in such a problem situation as evaluating $27^3 - 9^3$. The teacher must be aware of it, and use it constantly to achieve the "flow" we are talking about. Such a property may be useful to students at many different points of their study. However, its manner of presentation and the terminology used in discussing it are matters for the teacher to decide, based on the maturity of the students.

SUGGESTED READINGS FOR THE TEACHER

1. Andree, R. V. Selections from Modern Abstract Algebra. Henry Holt and Company, New York, 1958.
2. Apostol, T. M. Calculus, Vol. 1. Blaisdell Publishing Co., New York, 1961.
3. Beckenbach, E., and Bellman, R. An Introduction to Inequalities. Random House, Inc., New York, 1961.
4. Bell, E. T. The Last Problem. Simon and Schuster, New York, 1961.
5. Birkhoff, G., and MacLane, S. A Survey of Modern Algebra, Revised Edition. The Macmillan Co., New York, 1953.
6. Boehm, George A.W. The New World of Math. The Dial Press, New York, 1959.
7. Bruner, Jerome S. On Knowing: Essays for the Left Hand. Harvard University Press, Cambridge, Mass., 1962.
8. Bruner, Jerome S. On Learning Mathematics. The Mathematics Teacher, Vol. LIII, No. 8, (December, 1960), pages 610-619.
9. Bruner, Jerome S. The Process of Education. Harvard University Press, Cambridge, Mass., 1960.
10. Commission on Mathematics of the College Entrance Examination Board. Program for College Preparatory Mathematics. (report of the Commission), and Appendices (published as a separate volume). College Entrance Examination Board, New York, 1959.

11. Courant, R. and Robbins, H. What is Mathematics? Oxford University Press, New York, 1941.

12. Courant, R. Differential and Integral Calculus, Vol. 1. Interscience Publishers, Inc., New York, 1954.

13. Dubisch, R. The Nature of Number. The Ronald Press Co., New York, 1952.

14. Duren, W. L. Jr. The Maneuvers in Set Thinking. The Mathematics Teacher, Vol. LI, No. 5, (May, 1958), pages 322-335.

15. Gibb, G., Jones, P., and Junge, C. Number and Operation. The Twenty-Fourth Yearbook of The National Council of Teachers of Mathematics, Washington, D.C., 1959. Chapter 2.

16. Haaser, N.B., La Salle, J. P. and Sullivan, J. A. Introduction to Analysis, Vol. 1. Ginn and Co., Boston, 1959.

17. "Insights into Modern Mathematics." The Twenty-Third Yearbook of The National Council of Teachers of Mathematics. Washington, D.C., 1957.

18. Jones, Burton W. Miniature Number Systems. The Mathematics Teacher, Vol. LI, No. 4, April, 1958, pp. 226-231.

19. Kelly, John L. Introduction to Modern Algebra. D. Van Nostrand, Inc., Princeton, New Jersey, 1960.

20. Kemeny, John G., Snell, J. L. and Thompson, G. L. Introduction to Finite Mathematics. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1957.

21. Klein, Felix. Elementary Mathematics from an Advanced Standpoint. (Gottingen, 1908, Vol. 1 reprinted). Dover Publications, New York, 1952. pp. 77-85.

22. Levi, Howard. Elements of Algebra. Chelsea Publishing Co., New York, 1954.

23. Niven, Ivan. Numbers: Rational and Irrational. Random House, Inc., New York, 1961.

24. Sawyer, W. W. A Concrete Approach to Abstract Algebra. W. H. Freeman and Co., San Francisco, 1959.

25. Sawyer, W. W. What is Calculus About? Random House, Inc., New York, 1961.

26. School Mathematics Study Group. Studies in Mathematics: Vol. 1. Luce, R. D. Some Basic Mathematical Concepts. 1959. Vol. III. Haag, Vincent H. Structure of Elementary Algebra. 1959. New Haven, Conn.

27. Tarski, Alfred. Introduction to Logic. Oxford University Press, New York, 1956, Ch. 1.

28. Van Engen, H. and May, K. O. Relations and Functions. The Twenty-Fourth Yearbook of the National Council of Teachers of Mathematics. Washington, D.C., 1959. Chapter 3.

29. Whitehead, A. N. The Aims of Education and Other Essays. The Macmillan Co., New York, 1929. Chs. 1, 2, 3, 6.

30. Whitehead, A. N. An Introduction to Mathematics. Henry Holt and Co., New York, 1911.

31. Whitehead, A. N. Science and the Modern World. The Macmillan Co., New York, 1939, Ch. 2.

32. Young, J. W. Lectures on Fundamental Concepts of Algebra and Geometry. The Macmillan Co., New York, 1939.

FILMS AND FILMSTRIPS

The following are only suggestions for enrichment materials. This is a sampling, and not a complete list, of what is available. New materials are constantly being produced. Teachers can keep themselves up to date on these materials by reading the reviews and advertisements in the professional journals. For a complete listing of educational films and filmstrips, kept up to date by supplements, see Educational Film Guide and Filmstrip Guide, both published by the H. W. Wilson Company, 950 University Avenue, New York 52, N.Y.

An index of producers and distributors is to be found at the end of this list. The following categories correspond to chapters in the text.

1. Rational Numbers

Films:

Axioms (IFB). 1 reel 11 min color. Pictographic equations are utilized to demonstrate addition, subtraction, multiplication and division axioms.

Natural Numbers, Integers and Rational Numbers (Modern). 1-1/2 reels 29 min b-w. Reviews the properties of natural numbers in terms of comparison of magnitudes, combination (addition), and multiplication.

Addition and Subtraction of Rational Numbers (Modern). 2 reels 31 min b-w. Presents examples of the sums and differences of positive and negative numbers.

Multiplication of Rational Numbers (Modern). 2 reels 31 min b-w. Develops the multiplication of signed numbers for a variety of conditions.

Filmstrips:

The Closure, Commutative, and Associative Properties of Real Numbers (SVE). 50 fr color.

The Distributive Property (SVE). 40 fr color.

Identity and Inverse Properties (SVE). 55 fr color.

Order Properties (SVE). 51 fr color.

2. Equations and Inequalities

Films:

Linear Equations in One Unknown (Modern). 2 reels 30 min b-w. Two word problems are solved in detail followed by a discussion of equivalent equations and extraneous roots.

3. Systems of Linear Equations: Word Problems

Films:

Graphing Linear Equations (Coronet). 1 reel 10 min color or b-w. Line graphs represent relationship between variables. Shows how the values of constant terms and varying coefficients affect graphs.

Developing and Solving Linear Equations (Modern). 2 reels 30 min b-w. Exemplifies the need for establishing consistent conditions on X in the setting and solution of word problems involving linear equations.

Solving Simultaneous Linear Equations (Modern). 1-1/2 reels 29 min b-w. Graphical solutions are shown and analytical solutions are carried out to demonstrate the power of algebraic methods.

More Solutions of Linear Equations (Modern). 2 reels 30 min b-w. Applies good strategy in problem solving to four examples, and checks each solution.

Determinants and Cramer's Rule (Modern). 2 reels 30 min b-w. A review of second order determinants is obtained by solving a pair of linear equations.

Determinants of Any Order (Modern). 2 reels 30 min b-w. The general theorem and method of expanding a determinant by the minors of a row or column is shown.

4. Factored Forms

Films:

Beginning Algebra II (Norwood). 3 films. Lecture 1: Special Products and Factoring. 2 reels 30 min b-w. Explains and demonstrates the removal of a common factor. Lecture 2: Special Products and Factoring 2 reels 30 min b-w. Describes and demonstrates horizontal multiplication of binomials, factoring a general trinomial, and factoring perfect square trinomials. Lecture 3: Special Products and Factoring. 2 reels 30 min b-w. Demonstrates how to factor the difference of two perfect squares and a quantity squared.

Special Products and Factoring (Modern). 2 reels 31 min b-w. Shows that factoring is rewriting a mathematical expression as the product of two or more factors.

5. Fractions

Films:

Beginning Algebra II (Norwood). 2 films. Lecture 5: Fractions, Fundamental Operations. 2 reels 30 min b-w. Explains and demonstrates the fundamental operations of multiplication and division of fractions. Lecture 6: Fractions, Fundamental Operations. 2 reels 30 min b-w. Explains and demonstrates the fundamental operations of addition and subtraction of fractions.

Algebraic and Complex Fractions (Modern). 1-1/2 reels 29 min b-w. The method of simplifying complex fractions by dealing with the numerator and denominator separately is presented.

6. Quadratic Equations with Rational Roots

Films:

Beginning Algebra II (Norwood). 5 films. Lecture 4: Special Products, Factoring, and Quadratic Equations. 2 reels 30 min b-w. Explains complete factoring; quadratic equations solved by factoring; and the application of quadratic equations. Lecture 7: Fractional Equations. 2 reels 30 min b-w. Discusses and demonstrates the general solution of fractional equations. Lecture 8: Fractional Equations. 2 reels 30 min b-w. Explains the application of fractional equations. Lecture 15: Quadratic Equations. 2 reels 30 min b-w. Explains incomplete quadratics and demonstrates the solution of problems by factoring. Lecture 16: Quadratic Equations. 2 reels 30 min b-w. Explains the application of quadratic equations.

Quadratic Equations (Modern). 1-1/2 reels 29 min b-w. The standard form of a quadratic expression is analyzed to find the various conditions that make it solvable.

Introduction to Quadratic Equations (Modern). 2 reels 30 min b-w. The solution of word problems leading to quadratic equations is developed.

7. Formulas; General Conclusions

Films:

Formulas in Mathematics (IFB). 1 reel 10 min color. Based on the formula "distance equals rate multiplied by time" (d = rt). This mathematical law is graphically and pictorially illustrated.

Filmstrips:

Equations and Formulas (JH). 55 fr b-w.

8. Irrational Numbers

Films:

Beginning Algebra II (Norwood). 3 films. Lecture 12: Powers, Roots, and Radicals. 2 reels 30 min b-w. Discusses fractional exponents, the extraction of square roots, and use of the Pythagorean theorem. Lecture 13: Powers, Roots, and Radicals. 2 reels 30 min b-w. Demonstrates the simplifying and combining of radicals. Lecture 14: Powers, Roots, and Radicals. 2 reels 30 min b-w. Discusses rationalizing denominators; explains and demonstrates the multiplication and division of radicals.

Radicals and the Real Number System (Modern). 1-1/2 reels 29 min b-w. Presents radicals of negative numbers and gives many examples of methods for handling them.

Roots of High Order (Modern). 1-1/2 reels 28 min b-w. Shows how radicals of higher order than two can occur as solutions of certain equations.

General Methods for Solving Quadratic Equations (Modern). 1-1/2 reels 29 min b-w. The quadratic formula is derived and applied in examples.

Solving Problems with the Quadratic Formula (Modern). 1-1/2 reels 29 min b-w. Demonstrates the use of the quadratic formula.

9. Functions—Graphs—Variations

Films:

Beginning Algebra II. Lecture 9: Ratio, Proportion, and Variation (Norwood). 2 reels 30 min b-w. Explains the general concepts of ratio, proportion, and variation, and demonstrates their use in the solution of problems.

Variation—A Lesson in Reading (Modern). 1-1/2 reels 26 min b-w. Mathematical expressions for the simple relationship between two quantities are presented and the significance of constants of proportionality discussed.

Filmstrips:

Graphs (JH). 3 fs 55 fr each b-w. (1) Graph Uses (2) Plotting Graphs (3) Analytic Geometry.

10. Exponents and Logarithms

Films:

Logarithms and the Slide Rule (IFB). 8 films on the meaning and use of logarithms. Lesson I - 2 reels 30 min b-w. Gives introductory and historical information about logarithms and the slide rule. Lesson II - 2 reels 30 min b-w. Places emphasis on the mantissa determined by the sequence of significant digits. Lesson III - 2 reels 30 min b-w. Explains the method of computing mantissas. Lesson IV - 2 reels 30 min b-w. The method of looking up logarithms and antilogarithms is explained. Lesson V - 2 reels 30 min b-w. Presents a detailed explanation of the method of designing the slide rule. Lesson VI - 2 reels 30 min b-w. Explains division, using the C and D scales. Lesson VII - 2 reels 30 min b-w. Explanation and problem solving with the CI scale. Lesson VIII - 2 reels 30 min b-w. Demonstrates finding the area of a circle using the A and B scales and special marking for π.

Working with Positive and Negative Exponents (Modern). 2 reels 30 min b-w. A variety of examples illustrate the operation of both positive and negative exponents.

Using Fractional and Rational Exponents (Modern). 1-1/2 reels 28 min b-w. The properties of rational numbers are presented for fractional exponents of real numbers.

Equations with Unknowns in the Exponents (Modern). 2 reels 30 min b-w. Detailed presentation of the general nature of the logarithmic relations.

Using Logarithms to Solve Equations (Modern). 1-1/2 reels 28 min b-w. The method of using logarithm tables to find the mantissa, including the use of linear interpolation, is shown step by step.

Nature of Logarithms (Modern). 2 reels 31 min b-w. Develops the nature of logarithms on the basis of powers of ten and the properties of exponents.

Using Logarithms in Problems (Modern). 2 reels 31 min b-w. The logarithm of a number is defined for any arbitrary base.

Computing Logarithms from Arithmetic and Geometric Series (Modern). 1-1/2 reels 29 min b-w. An explanation of the computation of logarithm tables using the ratio method and interpolation.

Filmstrips:

Exponents and Logarithms (JH). 55 fr b-w.

11. Measurement in Right Triangles

Films:

Right Triangles and Trigonometric Ratios (Modern). 1-1/2 reels 29 min b-w. Illustrated definitions of sine, cosine, and tangent of an angle as ratios of the sides of a triangle having one right angle.

Using Sines, Cosines and Tangents (Modern). 1-1/2 reels 29 min b-w. A series of examples reviews the values of the sine, cosine, and tangent functions for 30, 45 and 60 degree angles.

Cosecant, Secant and Cotangent (Modern). 1-1/2 reels 27 min b-w. The cosecant, secant and cotangent functions are defined and shown to be the reciprocals of the sine, cosine, and tangent functions.

Beginning Algebra II (Norwood). 2 films. Lecture 10: Numerical Trigonometry. 2 reels 31 min b-w. Explains the basic functions and demonstrates simple applications of numerical trigonometry. Lecture 11: Numerical Trigonometry 2 reels 30 min b-w. Discusses and demonstrates the simple application of interpolation to practical problems.

Filmstrips:

Angular Measurement (JH). 55 fr b-w.

12. Elements of Coordinate Geometry

Films:

Algebra of Points and Lines (Modern). 2 reels 30 min b-w. The elements of simple analytic geometry are presented through a study of points and lines.

13. The Quadratic Function

Films:

Equations and Graphs of the Parabola (Modern). 2 reels 30 min b-w. A step-by-step development through the process of computing and plotting coordinate pairs for the parabola.

Complex Numbers and Roots of Equations (Modern). 2 reels 30 min b-w. Systems of quadratic equations and the nature of their roots are discussed.

Imaginary and Complex Numbers (Modern). 1-1/2 reels 29 min b-w. Methods of plotting complex numbers, using axes of reals and of imaginaries, are presented in some detail.

14. Equations of the Second Degree and Their Graphs

Films:

Locus (Knowledge). 1 reel 10 min. b-w. The concept of locus is visualized and explained by means of animation and photographic motion.

Introduction to Graphs of Equations (Modern). 2 reels 30 min b-w. Presents graphical interpretation of quadratic systems.

Graphs of Quadratic Equations (Modern). 1-1/2 reels 29 min b-w. Explains the nature and roots of equations for circles, ellipses and parabolas.

The Circle (Knowledge). 1 reel 10 min b-w. Presents important phases of radii, diameters, chords, tangents, secants, arcs, and central angles.

Hyperbola, Ellipse and Circle (Modern). 2 reels 30 min b-w. Shows the two kinds of hyperbolas, symmetric about the 45 degree axes and about the 90 degree axes.

15. The Sine and Cosine Functions; Oblique Triangles

Films:

Angles (Knowledge). 1 reel 10 min b-w. Develops an understanding of all of the various types of angles and their relationship to each other.

Large Angles and Coordinate Axes (Modern). 2 reels 30 min b-w. Defines and describes large angles and their trigonometric values.

Law of Sines (Modern). 2 reels 30 min b-w. The equation for the Law of Sines is derived and used to solve a problem.

Law of Cosines (Modern). 2 reels 30 min b-w. A step-by-step development of the Law of Cosines is shown and then applied.

Law of Tangents (Modern). 1-1/2 reels 28 min b-w. Describes the generation of a circle, ellipse, parabola, and hyperbola by sections through a cone.

Pythagorean Theorem: The Cosine Formula (Coronet). 1 reel 6 min b-w. Shows how the Pythagorean Theorem may be applied to acute triangles in addition to right triangles, and how this leads to the cosine formula.

Filmstrips:

Vectors (JH). 55 fr b-w.

Trigonometry (JH). 55 fr b-w.

16. Trigonometric Equations and Identities

Films:

Pythagorean Theorem (Knowledge). 1 reel 10 min b-w. Shows examples to prove the theorem and stresses its basic importance.

Eight Fundamental Trigonometric Identities (Modern). 1-1/2 reels 28 min b-w. Examples illustrate the use of the fundamental trigonometric identities.

Working with Trigonometric Identities (Modern). 1-1/2 reels 29 min b-w. The eight fundamental trigonometric identities serve as a basis for analysis of more complicated identities.

Trigonometry of Large Angles (Modern). 2 reels 30 min b-w. Applies trigonometric functions in terms of R, X and Y to large angles.

17. Polynomials

Films:

Theory of Equations and Synthetic Division (Modern). 1-1/2 reels 29 min b-w. Extends the technique of plotting by "completing the square" to a variety of parabolas, using synthetic division for testing for roots of an equation.

Solution of Equations Beyond the Second Degree (Modern). 2 reels 31 min b-w. The method of synthetic division is applied to finding the roots of a 4th degree equation.

18. Sequences and Series

Films:

Progressions, Sequences and Series (Modern). 1-1/2 reels 29 min b-w. Develops the law of formation for arithmetic progression and for geometric progression, and discusses the way to find a series from known terms.

Infinite Series and the Binomial Expansion (Modern). 1-1/2 reels 28 min b-w. The binomial expansion is worked out in detail with examples.

19. Trigonometric Addition Formulas

Films:

Addition Formulas and DeMoivre's Theorem (Modern). 1-1/2 reels 28 min b-w. A survey of several applications of trigonometry to algebraic equations.

Double and Half Angle Formulas (Modern). 1-1/2 reels 28 min b-w. The half-angle formulas for sine, cosine, and tangent are derived and used to simplify another identity.

INDEX TO PRODUCERS AND DISTRIBUTORS

Coronet	Coronet Films, Coronet Building Chicago 1, Ill.

IFB	International Film Bureau 332 South Michigan Ave., Chicago 4, Ill.
JH	The Jam Handy Organization 2821 East Grand Blvd., Detroit 11, Mich.
Knowledge	Knowledge Builders Films, Visual Education Building, Floral Park, N.Y.
Modern	Modern Talking Picture Service 3 East 54th St., New York 22, N.Y.
Norwood	Norwood Films 926 New Jersey Ave., Washington 1, D.C.
SVE	Society for Visual Education, Inc. 1345 Diversey Parkway, Chicago 14, Ill.

SUGGESTED TWO-YEAR INTEGRATED COURSE IN ALGEBRA AND GEOMETRY

If it is administratively feasible we think that there is great virtue in teaching half of this algebra text in Grade 10 and the other half in Grade 11. In addition, the Weeks-Adkins: A Course in Geometry, Plane and Solid, would be taught, half in Grade 10 and half in Grade 11. A suggested integration would be as follows:

Grade 10 { Geometry—Chapters 1 to 15
Algebra—Chapters 1 to 12

Grade 11 { Geometry—Chapters 16 to 23
Algebra—Chapters 13 to 19

Many arguments can be adduced for this arrangement of the materials. We mention three of them:

1. The basic unity of mathematics and of mathematical thinking is well conveyed because the number and variety of illustrations of this unity and thinking are greatly increased.

2. The easier parts of both algebra and geometry come in Grade 10 and the harder parts in Grade 11. This makes the subject easier in Grade 10 and permits a more mature approach in Grade 11.

3. Both subjects are kept fresher and used in a greater variety of situations all the time so that, by the end of Grade 11, there is firm control of all the mathematics of Grades 9, 10, and 11 and a great feeling by the student that he knows the subject and that it makes sense to him.

On pp. 18-20 are outlines of a possible course for Grade 10

and of a course for Grade 11 based on the assumption that the student has in hand each of the two books, Weeks-Adkins: Second Course in Algebra with Trigonometry and Weeks-Adkins: A Course in Geometry, Plane and Solid. In the outlines, "A" refers to the algebra book, "G" refers to the geometry book. The outlines assume a forty-five to fifty minute class period for each of 180 days.

The reader should realize that these outlines represent only one of several possible sequences. The requirements of individual teachers or special school situations may make some modifications desirable. For example, in Grade 11 some schools may wish to include more of the trigonometry than is indicated in this outline. We also remind the reader that certain topics, such as those contained in Chapters 16, 19, and 21 in the geometry book are essentially equivalent to those in Chapters 11, 12 and part of 14 in the algebra book. The reason for the overlap is to make either book, when used by itself for a year, more flexible than it would be without these chapters. Obviously, in the integrated course we are about to outline, one would pick the content of each of these chapters from one or the other of the books, but not from both.

Whatever sequence one chooses for the integrated course, the arrangements of the materials in these two books complement one another so that the teacher who uses any reasonable modification of the outlines below can be assured that materials will not suddenly arise in the geometry book for which the algebraic preparation is inadequate, nor will materials appear in the algebra book for which the geometric preparation is inadequate. The latter cannot possibly occur because the arrangement of the algebra book recognizes the possibility of some schools wishing to use it immediately after the first course in algebra and during the year before the geometry.

One of the major questions on which schools might differ when using the integrated plan is whether to open the tenth grade course with algebra or with geometry. This, we think, can be decided best in the light of the kind of course the school has offered to its ninth graders. If a lot of geometry has occurred in that course—at least toward the end of it—then one may wish to start the tenth grade with algebra. We prefer to start it with geometry, as the outlines below indicate.

We believe that the question of maturity is extremely

important in planning this program. To emphasize our feelings about this we suggest that you turn to the Table of Contents of each book and note the titles of Chapters 17 to 23 in A Course in Geometry, Plane and Solid and the titles of Chapters 13 to 19 in the Second Course in Algebra with Trigonometry. In each of these chapters will be found topics that you would probably skip in Grade 10 but find quite teachable in Grade 11.

Outline of an Integrated Mathematics Course for Grade 10

(Forty-five to fifty minute class periods for 180 days)

"A" refers to Weeks-Adkins: Second Course in Algebra with Trigonometry

"G" refers to Weeks-Adkins: A Course in Geometry, Plane and Solid

	No. of days
1. G. - Chapters 1-5	
Basic Ideas	
Line Segments and Angles as Numerical Quantities	
Proof	
Triangles	
Constructions	18
2. A. - Chapters 1-4	
Rational Numbers	
Equations and Inequalities	
Systems of Linear Equations: Word Problems	
Factored Forms	15
3. G. - Chapters 6-10	
Perpendicular Lines and Planes	
Parallel Lines	
Polygons: Angle Relationships	
Quadrilaterals	
Parallel Lines and Planes	20
4. A. - Chapters 5, 6, 7	
Fractions	
Quadratic Equations with Rational Roots	
Formulas: General Conclusions	15
5. G. - Chapters 11-14 (through p. 238)	
Further Study of the Triangle	
Inequalities	
Ratio and Proportion	
Similar Figures	22
6. A. - Chapter 8	
Irrational Numbers	15

	No. of days
7. G. - Chapter 14 (pp. 239-256)	
Pythagorean Theorem Problems in Three Dimensions	15
8. A. - Chapters 9-10	
Function, Graphs, Variation Exponents and Logarithm	20
9. G. - Chapter 15	
Areas and Volumes	15
10. G. - Chapter 19	
Measurements in a Right Triangle	10
11. A. - Chapter 12	
Elements of Coordinate Geometry	15
	180

Outline of an Integrated Mathematics Course for Grade 11

(Forty-five to fifty minute class periods for 180 days)

"A" refers to Weeks-Adkins: Second Course in Algebra

"G" refers to Weeks-Adkins: A Course in Geometry, Plane and Solid

	No. of days
1. A. - Chapter 12	
Review of the Linear Function	8
2. A. - Chapter 13	
The Quadratic Function	26
3. G. - Chapters 17, 18	
Locus in General Geometry of the Circle	20
4. A. - Chapter 14	
Equations of the Second Degree and Their Graphs	15
5. A. - Chapters 10, 11, 15	
Review of logarithms and of right triangle trigonometry The Sine and Cosine Functions: Oblique Triangles	
G. - Chapter 19 (pp. 408-413)	
Review three-dimensional situations	25

	No. of days
6. G. - Chapter 20	
Measurements in a Circle Regular polygons	10
7. G. - Chapter 22	
Solids with Curved Surfaces, Areas and Volumes Review of Chapter 15	15
8. A. - Chapter 17	
Polynomials	26
9. A. - Chapter 16	
Trigonometric Equations and Identities	10
10. A. - Chapter 18	
Sequences and Series	25
	180

It will be noticed that Chapter 19 of the algebra text and Chapters 23 and 24 of the geometry text are not included in the outlines. Their presence in the books does not mean that they are essential parts of a course based on the books. Their presence merely gives the teacher greater leeway in planning a course. In both books there are obvious ways of modifying these outlines to compensate for classes of higher or lower abilities. The outlines seem to us to indicate maximum possible coverage for a 180-day year. Each chapter can be pruned to the level of a given class. The basic ideas of each chapter, however, taken all together, provide a sequence and development that we think is clear, coherent, and teachable.

One of the important points to be noted in the use of these outlines is that the sequence is not to be taken with complete literalness. For example, in items 1 and 2 of the Grade 10 outline there may very well come times in the development of the introductory phases of the geometry when the end of a class period finds one with a topic only partially developed and an inadequate foundation laid for an assignment in the geometry. In this situation it is very handy to have available the algebra book in which an assignment can be found that one can be reasonably sure the pupils can do. Greater variety and therefore presumably interest can also be achieved by assignments that frequently use both books. It will be clear from the outlines whether a couple of items are best done in the order written down or whether they

can be done concurrently. The good teacher, after a year of experience, will find many interesting variations possible in the techniques of making assignments and in the integration of the various aspects of the subject matter.

To illustrate, the first four items of the Grade 11 outline need not be done completely in the order in which they are given in the outline. We present below detailed daily assignments for the first 20 days of the course to illustrate how much variety can be achieved. The reader may be interested to compare these assignments with those given for Chapter 14 of the algebra book, on page 65 of this manual. As was done there, the notation under "Class Discussion" indicates the main topic for presentation or discussion on the given day, and the assignments listed are to be made on the given day for the following day. Thus, the assignment of pp. 275-276, Ex. 1-12 in the algebra book, listed for Day no. 1, is to be announced on Day no. 1 as the homework for Day no. 2. The division by days is not meant to be rigid—that is, the class discussion on a given day will probably also include discussion of the previous day's assignment and/or future assignments. An "overlapping" of topics is natural, and some of this overlapping will be found in the assignments themselves.

Day No.	Class Discussion	Assignments: Algebra	Assignments: Geometry
1	Summarize Chapter 12(A), putting coordinate geometry into the context of the course.	pp. 275-276, Ex. 1-12	—
2	Proofs of geometric theorems.	pp. 278-279, Ex. 2-6, 8, 9	—
3	The construction of the linear equation that defines a set of points. The intersection of two lines.	pp. 282-283, Ex. 1, 6, 7, 9, 11-14, 18, 22, 23, 26, 27 pp.285-286, Ex. 1-3, 9, 13	—
4	How to arrange systematically the solution to such problems as Ex. 10 at the top of p. 287	pp. 286-287, Ex. 5, 6, 10, 11 Read pp. 315-319	—
5	The concept of locus and sets of points. (A) p. 287, Ex. 1. The quadratic function, $y = ax^2 + bx + c$.	p. 287, Ex. 12 p. 287, Ex. 2 p. 316, Ex.1, 8-10, 14	—

Day No.	Class Discussion	Assignments: Algebra	Assignments: Geometry
6	Locus in general. Loci described by words and by equations. (G) pp. 333 ff.	p. 287, Ex. 7 pp. 319-320, Ex. 2, 3, 6, 7, 9	Read pp. 333-336
7	The parabola as a locus, (A) pp. 321-323. (G) pp. 336-337, Ex. 1-3, 5-12 (orally).	p. 288, Ex. 8 p. 324, Ex. 1, 4, 6, 9 p. 326, Ex. 1, 5	p. 337, Ex. 13, 17
8	The range of the quadratic function. (A) pp. 327-329.	p. 326, Ex. 2, 6 p. 330, Ex. 8, 10, 12, 14, 17, 18	p. 337, Ex. 14-17
9	Graphical solution of quadratic equations. Intersection of loci again. (G) pp. 345 ff.	p. 326, Ex. 7 p. 331, Ex. 11, 14, 15, 16 p. 334, Ex. 5	p. 338, Ex. 1, 2, 3 p. 346, Ex. 1
10	(G) p. 338, Ex. 1, 2, 3 will provide a motivation for inequalitites. Recall (A) pp. 20 ff., and Chapter 12 (G).	p. 335, Ex. 1, 4, 5, 7, 10, 11 p. 23, Ex. 7, 9, 11 p. 24, A Ex. 9, 11, 13	p. 202, Ex. 5 p. 338, Ex. 4, 8 p. 346, Ex. 2
11	Work period	—	—
12	Test - such as: (A) p. 24 (Chapter Test), Ex. 10, 14, 18. p. 327, Ex. 10. p. 331, Ex. 8, 9, 13, 17, 20. p. 335, Ex. 12	Read pp. 336-342 p. 339, Ex. 1-17 (odd) p. 341, Ex. 1-11 (odd)	—
13	Complex numbers	pp. 343-344, Ex. 1-53 (odd)	—
14	The nature of the roots of $ax^2 + bx + c = 0$. Sum and product of the roots of $ax^2 + bx + c = 0$.	p. 348, Ex. 1, 3, 5, 7, 9, 11, 13 pp. 350-351, Ex. 9, 10, 11, 13, 18, 19, 20, 23	p. 338, Ex. 5, 13
15	Work period. Correct homework and discuss (A) p. 352, Ex. 1, 2, 3.	pp. 351-352, Ex. 5, 7, 8, 11, 15, 19 pp. 352-353, Ex. 4, 5, 10	—
16	(G) pp. 341-343. Theorems 57 and 58.	p. 352, Ex. 14, 16, 21 p. 353, Ex. 6, 11	p. 338, Ex. 6, 14, 16 p. 344, Ex. 1-8

Day No.	Class Discussion	Assignments: Algebra	Assignments: Geometry
17	Formal constructions by means of intersection of loci. (G) pp. 345-346.	p. 353, Ex. 7, 13, 14	p. 344, Ex. 9 pp. 346-347, Ex. 3, 4, 10, 11
18	Ways of describing sets of points: (1) verbal. (G) p. 344, etc. (2) (A) pp. 353-355: formation of quadratic functions. (3) contrast (G) p. 339, Ex. 22 with (A) p. 324, Ex. 3.	pp. 355-356, Ex. 2, 4, 6, 8 p. 327, Ex. 9	p. 344, Ex. 10, 14 p. 347, Ex. 8
19	The general equation of the circle. Tangents to circles. (A) p. 375. pp. 378 ff.	p. 381, Ex. 1, 8, 10 p. 327, Ex. 10 p. 356, Ex. 11, 15, 16	p. 344, Ex. 12
20	The geometry of the circle - a survey of Chapter 18 (G).		

Notes on the above outline

This outline shows how the general plan of the course can open up the possibilities for discovery and generalizations by the pupils. The twenty lessons outlined here actually occurred, in approximately the manner and sequence indicated, with a better than average college preparatory class. The thoughtful reader will realize at once that if this sequence came about "naturally" in one class a quite different sequence might develop in another class. How far the teacher dares to go in letting the class determine its own speed and sequence depends on the quality of the class and the skill of the teacher.

It also depends on the attitude of the teacher. Many fine teachers would object to the degree of "mixing" that these outlines suggest. These teachers will find that the basic outlines of the integrated course as given on pages 18-20 of this manual provide a steadier sequence for both class and teacher and sufficient integration to make the sequence still effective and interesting. The end results will be just as good and the process certainly will not be as prosaic as the conventional year of algebra would make it. The teacher is again warned that when each assignment is taken from several different pages in two books, the problem of remembering which exercises you have

assigned from a particular page becomes acute and needs special attention, such as keeping check-off lists for each set of problems.

The four solutions which follow illustrate the ultimate effect of teaching the algebra and the geometry texts concurrently.

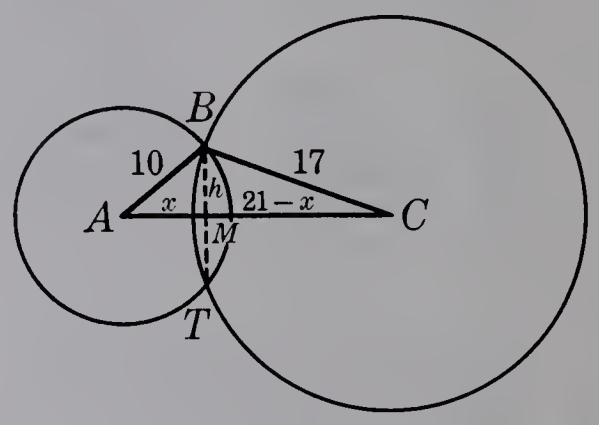
The problem: The radii of two circles are 17 in. and 10 in. and their centers are 21 in. apart. Find the length of their common chord.

First solution:

Let x = the length of AM

Then $x^2 + h^2 = 10^2$

and $(21 - x)^2 + h^2 = 17^2$



Subtracting,

$$-(21)^2 + 42x = 10^2 - 17^2$$

$$42x = 10^2 - 17^2 + 21^2$$

$$x = 6$$

$$\therefore h = \sqrt{10^2 - 6^2} = 8$$

$$\therefore BT = \underline{16} \text{ inches}$$

Second solution: (see figure for first solution)

We equate two expressions for the area of triangle ABC. The formulas used are $K = 1/2\ bh$ and $K = \sqrt{s(s - a)(s - b)(s - c)}$, where $s = 1/2(a + b + c)$.

$$\frac{21\,h}{2} = \sqrt{(24)(24 - 21)(24 - 17)(24 - 10)}$$

$$= \sqrt{(24)(3)(7)(14)}$$

Whence h = 8 and BT = 16 inches.

Third solution: (see figure for first solution)

From triangle ABC and the Law of Cosines, we have

$$17^2 = 10^2 + 21^2 - 2 \cdot 10 \cdot 21 \cdot \cos \angle A$$

$$\therefore \cos A = 3/5$$

But, from Triangle AMB, $\cos \angle A = \frac{AM}{10}$

$\therefore$ AM = 6. By the Pythagorean Theorem,

BM = 8

$\therefore$ BT = 16 in.

Fourth solution:

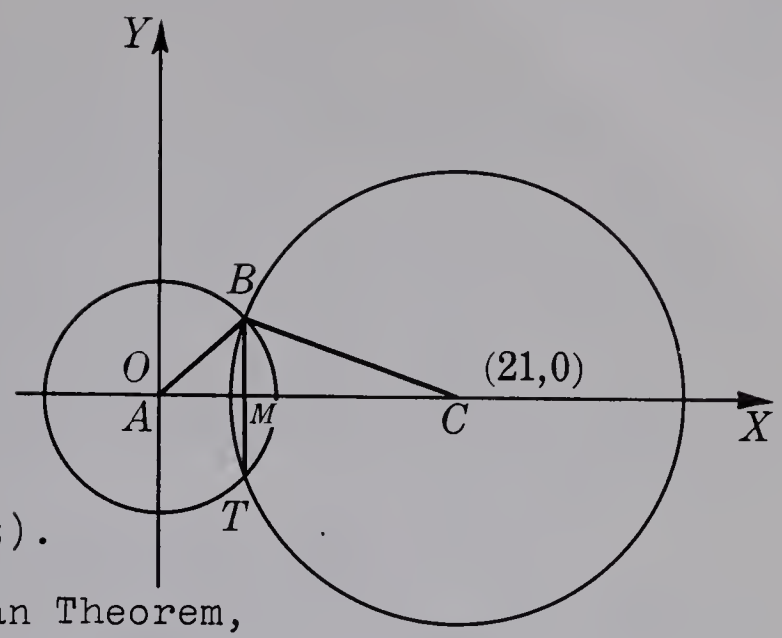
The circles may be placed as indicated on a cartesian coordinate system. Then the equations of the circles are

$$x^2 + y^2 = 100$$
$$(x - 21)^2 + y^2 = 289$$

Subtracting,

$$-42x = 441 = 189$$
$$x = 6$$

But x = 6 is the equation of the common chord, BT (See Example 2, pages 375-376 of the algebra text).



∴ AM = 6. By the Pythagorean Theorem,

BM = 8

and BT = 16 inches.

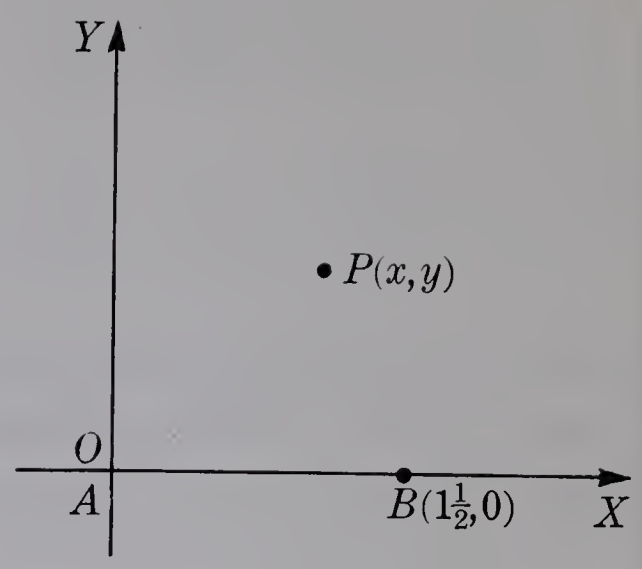
Obviously these four solutions have much in common. They are all basically dependent on the Pythagorean theorem. The first and fourth even look alike as far as symbolism goes. Only the words are different. We submit the opinion that by putting the student in a position where all these approaches are fresh and meaningful we illuminate the subject for him and emphasize, as no general statements can, the essential unity of the algebra, geometry, and trigonometry. And not the least compensation is that the pupils can really get excited by a discussion such as these four solutions epitomize.

Another illustration of the sort of thing that can happen in class when the algebra and geometry are taught concurrently is based on Exercises 22, 23, 24 on page 339 of the Geometry.

The first step in the sequence of maneuvers is to do one of them—the set of assignments suggests No. 22—when this chapter on Locus is studied. Then, after the ellipse and the hyperbola have been studied in Chapter 14 of the Algebra 2, come back to this page and assign No. 23 and No. 24 without making any remark that ties them to Chapter 14. You will find that about half the class will deal with them by analytic methods and the others will simply plot a lot of points. This gives an elegant opportunity to point out one of the great virtues of the analytic method, i.e., its easy and straightforward technique for handling a situation that may seem relatively complicated.

We illustrate by doing exercise 24: "Two points A and B are $1\frac{1}{2}$ inches apart. Plot the locus of a point which is always twice as far from A as it is from B."

Solution: Let us choose to place A at the origin and B on the x-axis. Then, if (x, y) represents the coordinates of any number of the desired set of points, the condition imposed by the words of the problem may be translated into the symbols:



$$PA = 2 \cdot \overline{PB}, \text{ and so we have}$$

$$\sqrt{x^2 + y^2} = 2\sqrt{(x - 1\tfrac{1}{2})^2 + y^2}$$

This reduces to $x^2 - 4x + y^2 = -3$, and if we complete the square in the x's, we have

$$(x^2 - 4x + 4) + y^2 = 1$$

or

$$(x - 2)^2 + y^2 = 1.$$

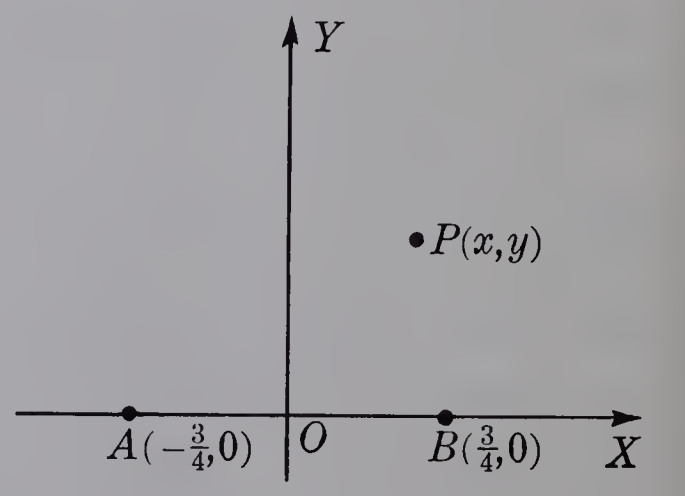
From this final equation we may immediately deduce that the locus is a circle with center $\frac{1}{2}$ inch from B on the extension of AB through B and having a radius of 1 inch.

If the teacher desires it, this problem provides an excellent opening into the subject of translation of axes. With any luck at all one member of the class will place the axes as we have in the solution above and another will place them so that the y-axis bisects the segment AB. In the latter case we have

$$\sqrt{(x + \tfrac{3}{4})^2 + y^2} = 2\sqrt{(x - \tfrac{3}{4})^2 + y^2}$$

Whence $x^2 - \frac{5}{2}x + y^2 = -\frac{9}{16}$

and so $(x - \frac{5}{4})^2 + y^2 = 1.$



This equation indicates that the locus is a circle of radius 1 in. and center at $(\frac{5}{4}, 0)$ which is $\frac{1}{2}$ in. from B on the extension of AB through B. The two equations

$$(x - 2)^2 + y^2 = 1 \quad (1)$$
$$(x - \tfrac{5}{4})^2 + y^2 = 1 \quad (2)$$

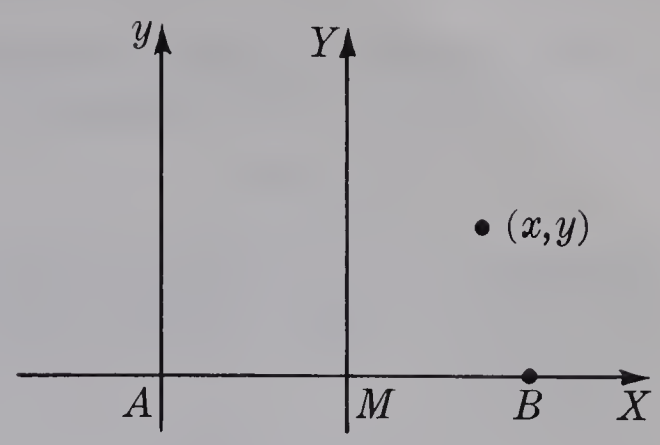
thus describe two circles which have equal radii and have their centers placed in identical positions with reference to AB. It should be possible, then, by a formal translation of the axes of (1) to change (1) so that it is identical with (2), or, of course, change (2) into (1). In (1), we translate the y-axis to Y so that M is

the midpoint of AB. Then $X = x - \frac{3}{4}$ and so, in ① we replace x by $X + \frac{3}{4}$ and y by Y, getting

$$(X + \tfrac{3}{4} - 2)^2 + Y^2 = 1$$

or $(X - \frac{5}{4})^2 + Y^2 = 1$.

But this is formally identical with ②.



From this discussion one can go on to problems like those on page 392 of the Algebra 2 book. Of course, we hasten to add that translation of axes is easy and nice, and not very important at this stage so the teacher is warned not to let enthusiasm carry the topic out of its proper minor spot in the course. In fact, we do not regard it as an essential part of the course. This is the reason why it does not appear in the book. We do think it a nice digression, however, if a good class creates the proper setting for its introduction. And it is offered here as one more example of the kind of thing that seems to happen in class more frequently when the algebra and geometry are taught concurrently throughout grades ten and eleven.

A broad picture of the interweaving of the main topics in Grades 10 and 11 may be obtained from the following tabular arrangement:

To Illustrate How Topics in Grades 10 and 11 are Interwoven

(Read from left to right)

ALGEBRA	GEOMETRY AND TRIGONOMETRY
Review of elementary operations, emphasizing postulates and the formation of algebraic expressions.	Definitions, postulates, proof. Congruence.
Quadratics with rational roots	Parallel and perpendicular lines and planes. Parallelograms, etc.
Ratio, proportion.	Proportional division. Similar triangles, mean proportional.
Square root, radicals, quadratics by completing square, formula.	Pythagorean theorem. Applications to three-dimensional figures.
Functions, graphs, variation. Exponents extended to rational numbers. Logarithms.	Trigonometric functions. Computation in right triangles. Applications to three-dimensional figures.

To Illustrate How Topics in Grades 10 and 11 are Interwoven

(Read from left to right)

ALGEBRA	GEOMETRY AND TRIGONOMETRY
Introduction to analytic geometry. Distance, slope, locus. Equation of straight line. The linear function.	Areas. Volumes of prisms, pyramids.
Graph of $f(x) = ax^2 + bx + c$. Definition of parabola. Extreme values of $f(x)$. Solution of $F(x) = 0$. Imaginary numbers. Use of discriminant.	Locus. Properties of the circle.
Second-degree equations in x and y. General equation of circle. Intersection of line and curve. Ellipse, hyperbola treated informally, noting intercepts, symmetry, asymptotes.	Review of logarithms, right-triangle trigonometry. Extension of definitions to general angle. Law of sines, law of cosines, area formula.
Polynomial functions, mainly cubic. Solution of cubic equations with rational roots. Synthetic division. Remainder theorem. Approximation to irrational root.	Regular polygons. Area of circle. Areas and volumes of cylinders, cones.
Arithmetic, geometric series. Limit of sum. Mathematical induction and binomial theorem.	Relationships between trigonometric functions. Equations, identities.

SUGGESTIONS FOR THE USE OF THE TEXT

Chapter 1. Rational Numbers

The purpose of Chapter 1 is to make the structure of algebra sufficiently explicit so that the student will not only realize (or recall) that the subject has a logical structure, but will find that structure simple enough and plain enough so that he will use its language throughout the course. We do not expect the student to learn the proofs of the theorems on pages 7 and 8. With a class of ordinary ability it may not even be desirable to go over the proofs of them. However, most students of algebra at this level are interested, in seeing that such statements as $(-a)(-b) = ab$ can be proved, and so we would suggest to a group that they read through these proofs slowly enough to follow them but not often enough to enable them to reproduce them. Or maybe

the teacher would prefer to run through them in class. In any case, don't get bogged down in this chapter. One shouldn't spend more than three days on it.

Chapter 2. Equations and Inequalities

Chapter 2 continues to review and strengthen the structural foundation of the subject. The nature of equality is highlighted by contrasting it with inequalities. Important technical terms and notations are assembled, worked with long enough to make them meaningful, and then left for future reference. Five days is a liberal amount of time for this chapter.

Chapter 3. Systems of Linear Equations

Word Problems

The relative simplicity of the technical skills called for by this chapter permits the teacher to focus attention on the problem of translating the relationships given by words into the equation forms useful for answering questions. In doing this, it is desirable to reiterate the distinction between relationships that are explicitly given by the words of the problem and those that are implied by the words of the problem.

There may be more word problems here than should be used at this time. How many of them to use depends very largely on the quality of the class. One problem of each "type" may well be enough for a good group. On the other hand, a college-capable group with a weak background may need to do nearly all of them. In any case you want to be sure that the group feels "comfortable" with this sort of exercise before going on.

Note on How to Make Assignments

This chapter illustrates well the problem of how to make assignments, particularly of material that is, at least partly, review and drill. One of the fundamental principles governing such assignments is that they should be as varied as possible. We assume that the first day of the sequence to be spent on this chapter has an assignment due, and this assignment we label "Day No. 1."

Day No. 1: { Read pages 27 to 30
page 31; 1, 7, 8, 12, 16, 17, 18 to 21

Day No. 2: { page 32 (A-2); 16 to 22, and
pages 32-33 (B); 1 to 5

Day No. 3: { page 33; 6, 7, and
pages 35-36; 1 to 6

Day No. 4: { page 33; 10,
page 36 (A-2); 1,
page 36 (B); 1, and
pages 40-41; 1, 3, 4, 5, 7

Day No. 5: { page 33; 11,
page 36 (A-2); 5, and
pages 40-41; 9, 11, 14, 16, 18, 20

Day No. 6: Chapter Review, pages 45-46

Day No. 7: Test on Chapter.

Of course, no two people are likely to come up with identical assignments, but the above is at least a plausible method. Some classes may require more practice. There are obviously plenty of unused problems to meet such a situation.

Chapter 4. Factored Forms

Each of the multitude of occasions in elementary algebra when a change of form is desirable usually involves the process of factoring. In this chapter we review factoring processes already dealt with in the first course in algebra, and add a few more, such as factoring the sum or difference of two cubes. The importance of patterns and the great usefulness of the distributive axiom receive particular attention.

In handling this chapter, it is again incumbent upon the teacher to assess quickly the skill of her class and to vary the time and effort spent in accordance with that skill. If there is a reasonable residue of skill from the first algebra course, then it is sometimes more fun to spend one day on Chapter 4; then start Chapter 5 on fractions and pick up additional practice work in Chapter 4 as the need for factoring skill arises in the problems of Chapter 5. In fact, we have had class situations where

Chapters 4, 5, and 6 could be done simultaneously. After all, the material is fundamentally review, and any variety in approach that takes you "out of the groove" is good and will be well received by the class. Just be careful that the work doesn't become superficial, and don't jump to conclusions about how much the students know without real evidence. A careless job on these chapters will spoil the rest of the book, because the skills involved are essential throughout the remainder of the course.

Chapter 5. Fractions

This chapter continues the review of elementary techniques and ideas. More and more the importance of words comes to the fore, and the usefulness of the axioms and theorems of the field of rational numbers becomes more apparent. Axiom M 4 (see page 4), which says that $a \cdot 1 = a$ for every a, occurs many times. In the book we have tended to say, for example, "multiply the numerator and the denominator by $x + 1$" instead of "multiply the fraction by 1 in the form $\frac{x+1}{x+1}$" simply because the latter phraseology seems a little clumsier than the former. We recommend that the teacher, in oral work, use both ways of speaking, but use the second more often than the first.

One of the outcomes of a proper handling of such technical situations as those on pages 93 and 94 is that the student will receive practice in writing his work down neatly and fully. Without disparaging the business of getting correct answers, one can still insist on neat and well-organized work. The habit thus created—or strengthened—will pay big dividends in the algebraic work of higher mathematics courses.

Chapter 6. Quadratic Equations with Rational Roots

The only idea in this chapter which may seem rather unfamiliar to the ordinary student is the possibility of transforming an equation into another equation which is not equivalent to the original equation. We discuss this on page 104, and the teacher should make certain students understand this discussion.

Pages 105, 106

More and more, as the algebraic sophistication of the student increases, he should be taught to look for sequences of maneuvers

which are more convenient and "efficient" than the usual sequence taught in elementary algebra. For example, to solve the equation, $\frac{x^2 + 1}{x + 2} = \frac{x}{3} + \frac{5}{x + 2}$, it is certainly "more orderly" to proceed as follows:

$$\frac{x^2 - 4}{x + 2} = \frac{x}{3} \quad \text{(Subtract } \frac{5}{x + 2} \text{ from both sides.)}$$

$$x - 2 = \frac{x}{3} \quad \text{(Reduce } \frac{x^2 - 4}{x + 2} \text{ to lowest terms.)}$$

$$3x - 6 = x$$

$$\therefore x = 3$$

Contrast the sequence just completed with the following:

$$\frac{x^2 + 1}{x + 2} = \frac{x}{3} + \frac{5}{x + 2}$$

$$3x^2 + 3 = x^2 + 2x + 15$$

$$2x^2 - 2x - 12 = 0$$

$$x^2 - x - 6 = 0$$

$$(x - 3)(x + 2) = 0$$

$$x = 3 \text{ or } -2$$

Then -2 must be ruled out since division by zero is undefined.

It is a very worthwhile exercise to go through these two pages of exercises orally in order to decide the most convenient and efficient sequence of steps to use. Sometimes the easiest first step is merely the usual one of multiplying by a common denominator. To illustrate the point of making changes in the sequence of attacking a problem, we give the first and sometimes the second step recommended in solving a few problems from Exercises (A-2) on page 106.

3. $\frac{4}{3x + 1} = \frac{7}{6x + 2}$

$\frac{3x + 1}{4} = \frac{2(3x + 1)}{7}$

$\therefore (3x + 1)[\frac{1}{4} - \frac{2}{7}] = 0$

$\therefore 3x + 1 = 0$

But this is impossible. Consequently, this equation does not have a solution.

18. $3(\frac{x + 1}{x - 1}) = \frac{2}{x - 4} + \frac{6}{x - 1}$

$\frac{3x + 3}{x - 1} = \frac{2}{x - 4}$

$3(x - 4) = 2$

etc.

14. $\frac{200}{x + 8} = \frac{200}{x} - \frac{5}{6}$

Dividing by 5,

$\frac{40}{x + 8} = \frac{40}{x} - \frac{1}{6}$

etc.

21. $\frac{x + 2}{x^2 - x - 6} = 3 - \frac{4}{x - 3}$

$\frac{5}{x - 3} = 3$

$x - 3 = 5/3$

etc.

Page 109 - Exercise 6

The conclusion called for in this problem is not at all obvious, and may prove to be quite exciting for the students. Since three consecutive positive integers may, in general, be represented by n, n + 1, and n + 2, where n is a positive integer, then we must have

$$(n+2)^2 = (n+1)^2 + n^2 \quad (A)$$
$$(n+2)^2 - (n+1)^2 = n^2$$
$$1(2n+3) = n^2$$
$$n^2 - 2n - 3 = 0$$
$$(n-3)(n+1) = 0$$
$$\therefore n = 3$$

We discard n = -1 since, by hypothesis, n stands for a positive integer.

Hence the only positive integer that satisfies (A) is 3.

Chapter 7. Formulas; General Conclusions

A genuine understanding of this chapter will pay tremendous dividends throughout the rest of the book. The increasing generality of the symbolism tends to force the student into an explicit description of his technical maneuvers. The variety of the problems helps to generalize procedures for handling the problems. The "if-then" pattern of thinking receives increasing emphasis. And, finally, the content of the chapter encourages the teacher to reiterate the kind of thinking illustrated by the following:

> The formula A = 1/2 bh says that, for all triangles it is true that the area is equal to one half the product of the base and altitude. If we wish to make a statement about any two triangles we may do so by using the subscript notation. Thus, to make the statement that if the altitudes of two triangles are equal then the ratio of their areas equals the ratio of their bases, we would write
>
> $A = 1/2\ bh$ (for all triangles)
> $A_1 = 1/2\ b_1 h_1$ (for any one member of the set of all $\triangle$)
> $A_2 = 1/2\ b_2 h_2$ (for any second member of the set of all $\triangle$)
> $h_1 = h_2$
> $$\therefore \frac{A_1}{A_2} = \frac{1/2\ b_1 h_1}{1/2\ b_2 h_2} = \frac{b_1 h_1}{b_2 h_2} = \frac{b_1}{b_2} \cdot \frac{h_1}{h_2} = \frac{b_1}{b_2} \cdot 1 = \frac{b_1}{b_2}.$$

Pages 116, 117

These two sets of problems will cause considerable difficulty if assigned casually and without initial work in class. We suggest that one class period be devoted to working out Exercises (A-1) and then Exercises (A-2) be used for homework. Some of the problems lend themselves nicely to showing how the sequence of maneuvers may be varied. For example, consider numbers 11, 19, and 25 in Exercises (A-1).

11. $ax + \frac{b}{a} = bx + \frac{a}{b}$

First Solution

$$(a-b)x = \frac{a}{b} - \frac{b}{a}$$

$$x = (\frac{a}{b} - \frac{b}{a})(\frac{1}{a-b})$$

$$x = \frac{a^2 - b^2}{ab(a-b)}$$

$$x = \frac{a+b}{ab}$$

Second Solution

$$a^2bx + b^2 = ab^2x + a^2$$

$$ab(a-b)x = a^2 - b^2$$

$$x = \frac{a+b}{ab}$$

19. $\frac{a-x}{a} + \frac{b-x}{b} = \frac{c+x}{c}$

First Solution

$$abc - bcx + abc - acx = abc + abx$$

$$(ab + bc + ac)x = abc$$

$$\frac{ab + bc + ac}{abc} = \frac{1}{x}$$

$$\frac{1}{x} = \frac{1}{c} + \frac{1}{a} + \frac{1}{b}$$

Second Solution

$$1 - \frac{x}{a} + 1 - \frac{x}{b} = 1 + \frac{x}{c}$$

$$1 = \frac{x}{a} + \frac{x}{b} + \frac{x}{c}$$

$$\frac{1}{x} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

25. First Solution

$$\begin{cases} \frac{a}{x} + \frac{b}{y} = a \\ \frac{b}{x} - \frac{a}{y} = 0 \end{cases}$$

$$\begin{cases} \frac{a}{x} = a - \frac{b}{y} & (1) \\ \frac{b}{x} = \frac{a}{y} & (2) \end{cases}$$

Dividing (1) by (2), we have:

$$\frac{a}{b} = (a - \frac{b}{y}) \cdot \frac{y}{a}$$

$$\frac{a}{b} = y - \frac{b}{a}$$

$$y = \frac{a}{b} + \frac{b}{a} = \frac{a^2 + b^2}{ab}$$

$$x = \frac{y}{a} \cdot b = (\frac{1}{b} + \frac{b}{a^2})b = 1 + \frac{b^2}{a^2}$$

$$x = \frac{a^2 + b^2}{a^2}$$

Second Solution

$$\begin{cases} \frac{a}{x} + \frac{b}{y} = a \\ \frac{b}{x} - \frac{a}{y} = 0 \end{cases}$$

$$\begin{cases} \frac{a^2}{x} + \frac{ab}{y} = a \qquad (1) \\ \frac{b^2}{x} - \frac{ab}{y} = 0 \qquad (2) \end{cases}$$

Adding (1) and (2), we have:

$$(a^2 + b^2)\frac{1}{x} = a^2$$

$$x = \frac{a^2 + b^2}{a^2}$$

etc.

Third Solution

$$\begin{cases} \frac{a}{x} + \frac{b}{y} = a \\ \frac{b}{x} - \frac{a}{y} = 0 \end{cases}$$

Let $R = \frac{1}{x}$

$$S = \frac{1}{y},$$

$$\begin{cases} aR + bS = a \\ bR - aS = 0 \end{cases}$$

$$R = \frac{aS}{b}$$

$$\frac{a^2 S}{b} + bS = a$$

$$S = a \div (\frac{a^2}{b} + b)$$

$$= a \cdot \frac{b}{a^2 + b^2}$$

$$S = \frac{ab}{a^2 + b^2} = \frac{1}{y}$$

$$y = \frac{a^2 + b^2}{ab}$$

etc.

Page 119

The word "ratio" should rapidly become a part of the students' working vocabulary if it has not already. The statement that the ratio of the lengths of two line segments, a and b, is 2 : 3 should immediately suggest the paraphrase, $\frac{a}{b} = \frac{2}{3}$, or the introduction of the notation, a = 2m, b = 3m. Thus a problem like the following may be handled in two ways.

Problem. If 3P = 2Q, state the ratio $\frac{P}{P + Q}$

First solution:

If $3P = 2Q$

then $\frac{Q}{P} = \frac{3}{2}$

$$\frac{Q}{P} + 1 = \frac{3}{2} + 1$$

$$\frac{Q + P}{P} = \frac{3 + 2}{2}$$

Whence $\frac{P}{P + Q} = \frac{2}{5}$

Second Solution:

If $3P = 2Q$

then $\frac{P}{Q} = \frac{2}{3}$

If we let $P = 2m$

then $Q = 3m$

and so $\frac{P}{P + Q} = \frac{2m}{2m + 3m} = \frac{2}{5}$

Pages 133-136

Although most beginning algebra courses include some work on irrational numbers, some classes do not study them during the first year. Therefore, these Cumulative Review Exercises are placed before the chapter on irrational numbers and may be used as a summary of Chapters 1 to 7. They cover quite thoroughly everything in these chapters, and we feel that good work on these exercises means that the pupil is ready to go on. We suggest that you check the time needed by your class to complete one set of review exercises and then modify the other groups of exercises to conform with the time you have available. We have found that an ordinary college preparatory group needs sixty minutes to comfortably complete one of the groups.

Another possibility, with a class that seems to be extremely well trained, is to start the course with these Cumulative Review Exercises and then go right on into chapter 8.

Chapter 8. Irrational Numbers

The general plan of this chapter is as follows. It opens with the familiar concepts of squares and square roots and uses these concepts to introduce square roots that are not exact. Attention is then focussed on the problem of expanding the number system to include the irrational numbers and meaning is given to such symbols as $\sqrt{3}$ by the discussion on pages 143-146. Ample opportunity to practice the fundamental operations with the elements of this expanded number system is then provided (pp. 148 to 156). Next, their use in solving the quadratic equation is exploited as an immediate and far-reaching application. The brief study of irrational expressions (pages 167-169) foreshadows the more general discussions of Chapter 9, and this treatment of irrational expressions is also connected with the section on irrational equations which follows it and completes the chapter. The capable and curious and unhurried student can then reach a satisfying climax with the proof that $\sqrt{3}$ is irrational. (See sidelight 7, pages 176, 177)

Page 139

The teacher should be aware of the fact that repeating decimals are representations of rational numbers and that the rational form can be obtained by using the formula for the limit of the sum of

an infinite geometric series. (See pages 508-510 of the text.) Thus, 3.4166 . . . may be written 3.41 + .006 + .0006 + .00006 - ... Then the limit of the sum of the infinite series whose first term is .006 and whose common ratio is 0.1 is given by the form $\frac{.006}{1 - 0.1} = \frac{1}{150}$, and so 3.4166..... may be written as $3\frac{41}{100} + \frac{1}{150}$ $= 3\frac{5}{12}$.

Page 140

The discussion of significant figures is concise and clear enough, we hope, that students can understand it. However, the teacher may find it useful to know, in addition to the discussion in the text, that figures which are necessary to determine the relative error of a measurement are significant. The relative error is the possible error per unit of measure. Thus, if a measurement is given as 6.2 then the unit of measure is tenths, the possible error is .05, and the relative error is $\frac{0.05}{6.2} = \frac{1}{124}$ or about 0.8%. The numeral 6.2 contains two significant figures. If the measurement is given as 6.200, then the possible error is .0005 and the relative error is $\frac{.0005}{6.200} = \frac{1}{12400}$. The numeral, 6.200, contains four significant figures. But if the measurement is given as .062, then possible error is .0005, the relative error is $\frac{.0005}{.062} = \frac{1}{124}$ and the number of significant figures is two.

Page 141

We do not consider that an algorithm for extracting a square root is a very useful part of the course, and therefore we have not included one. There is one in First Course in Algebra. We give here the one based on the identity $(a + b)^2 = a^2 + 2ab + b^2$.

$$\begin{array}{r|l} & a + b \\ \hline & \sqrt{a^2 + 2ab + b^2} \\ & a^2 \\ \hline 2a & 0 + 2ab + b^2 = b(2a + b) \\ + b & 2ab + b^2 \\ \hline 2a + b & 0 \end{array}$$

(A numerical example illustrating this may be found at the top of the next page.)

$$\begin{array}{rl|l} & 9\ \ 7 = 90 + 7 = a + b & \\ & \sqrt{94\ 09} = a^2 + 2ab + b^2 & \\ & 81 \quad\ = a^2 & \\ 2a = 180 & \overline{13}\ 09 = 2ab + b^2 = b(2a + b) & \\ +\ b = +7 & 13\ 09 & \\ \hline 2a + b = 187 & \quad 0 & \end{array}$$

This method can be extended to larger numbers and decimals. The following is an example:

$$\begin{array}{rl|l} & \underline{1\ \ 5.\ 7} & \\ & \sqrt{2\ 46.49} & \\ & 1 & \\ 2 \times 10 = 20 & 1\ 46 & \\ \underline{+5} & 1\ 25 & \\ 25 & & \\ 2 \times 150 = 300 & 21\ 49 & \\ \underline{+7} & 21\ 49 & \\ 307 & \quad 0 & \end{array}$$

Tables of square roots, and later on, logarithms, seem to us the way to handle this arithmetic in this text.

Pages 143-145

With a very good class the teacher may wish to add to the discussion here the proof given in Sidelight 7 at the end of the chapter which shows that $\sqrt{3}$ is irrational. Any further technical detail seems of doubtful value to us at this stage. A very great number of good discussions on irrational numbers are available in the literature. We list a few of them here for the convenience of the teacher.

Niven, Ivan. Numbers: Rational and Irrational. Random House, New York, 1961. One of the monograph projects of the School Mathematics Study Group.

Niven, Ivan. Irrational Numbers. No. 11 of the Caries Mathematical Monographs of the Mathematical Association of America. John Wiley and Sons, New York, 1956. A more sophisticated treatment than the preceding reference.

Levi, Howard. Elements of Algebra. Chelsea Publishing Company, New York.

Ringenberg, Lawrence A. A Portrait of 2. Washington, D.C.: National Council of Teachers of Mathematics.

Pages 148, 149

Even students who are mature enough to be studying this book need a lot of practice on exercises of this sort. Do many of

these problems orally and then assign the same problems for written work. This double exposure is essential to the development of reasonable facility in such manipulations.

Page 150

The pupil may quite properly be reminded at this point that if $ab + ac = (b + c)a$ is assumed to hold for irrational numbers, then it includes such cases as $5\sqrt{3} + 7\sqrt{3} = (5 + 7)\sqrt{3} = 12\sqrt{3}$.

Page 152

If a student has trouble understanding the process of division by $\sqrt{a} \pm \sqrt{b}$, a variation from the language used in the book may help. Thus, we may discuss Example 1, Divide $\sqrt{2}$ by $2 - \sqrt{2}$, in the following way. If we write the problem in the form $\frac{\sqrt{2}}{2 - \sqrt{2}}$ the desired goal may be brought to focus as follows: By what must we multiply $2 - \sqrt{2}$ so that the product will be a rational number? (Answer: $2 + \sqrt{2}$). Hence, because we must multiply the fraction by 1 in order not to change its value, we choose $\frac{2 + \sqrt{2}}{2 + \sqrt{2}}$ as a useful form of 1; etc.

Page 157

A quite useful and reasonable way to motivate the expansions of the number system is to base each expansion on the need for greater generality in the solution of equations. Thus, we added negative numbers in order to make the statement, "If $a + x = b$ then $x = b - a$," generally true. "If $ax = b$ then $x = \frac{b}{a}$" led us to rationals, and now "If $x^2 = b$, $(b \geqq 0)$, then $x = \pm\sqrt{b}$" makes it necessary to introduce irrational numbers. Finally (in Chapter 13) we will introduce imaginary numbers, so that $x = \pm\sqrt{b}$ may make sense even when $b < 0$. This is the last expansion of the number system that is necessary.

Pages 158, 159

Do not present the method of "completing the square" as if the procedure given in the book were the only one. We agree with most people that it is the simplest technique and that more than one procedure is not necessary. However, a digression in class along the lines indicated below can be fun and is somewhat illuminating, because it enables the teacher to talk in more general terms than when discussing one procedure only.

Solve the equation $2x^2 - 3x - 10 = 0$ by the method of completing the square.

$$2x^2 - 3x - 10 = 0$$
$$2x^2 - 3x = 10$$

Since, in any perfect square trinomial, $ax^2 + bx + c$, ax^2 and c must be perfect squares, we multiply by 2:

$$4x^2 - 6x = 20$$

Let k^2 represent the number to be added.

Then $$4x^2 - 6x + k^2 = 20 + k^2$$

Since $(2x - k)^2$ must be identically equal to $4x^2 - 6x + k^2$:

$$4x^2 - 4kx + k^2 = 4x^2 - 6x + k^2$$

then $$-4kx = -6x$$

and $$k = \frac{3}{2}$$

Whence $$k^2 = \frac{9}{4}$$

then $$(2x - \tfrac{3}{2})^2 = 20 + \frac{9}{4} = \frac{89}{4}$$

and $$2x - \frac{3}{2} = \pm\frac{1}{2}\sqrt{89}$$

$\therefore$ $$x = \frac{3 \pm \sqrt{89}}{4}$$

Whether you bring this procedure in now or wait until the pupils have reasonable skill and facility with <u>one</u> method depends on the quality of the class and on your own feeling about their readiness for greater generality.

The general technique of creating two expressions that are identical in order to compare the coefficients of like powers of the variable is also applicable to the problem of finding an exact form for a square root of such expressions as $54 - 20\sqrt{2}$. Here the underlying point is that if two numbers $a + b\sqrt{c}$ and $x + y\sqrt{c}$ are equal, then $a = x$ and $b = y$. The sequence of steps follows:

If $54 - 20\sqrt{2}$ has a square root of the form $a + b\sqrt{2}$,

then $$(a + b\sqrt{2})^2 = 54 - 20\sqrt{2}$$
$$a^2 + 2ab\sqrt{2} + 2b^2 = 54 - 20\sqrt{2}$$
$$(a^2 + 2b^2) + 2ab\sqrt{2} = 54 - 20\sqrt{2}$$

Then $$a^2 + 2b^2 = 54$$

and $$ab = -10$$

$$\therefore\ a = 2, -2 \text{ and } b = -5, 5.$$

Hence the square roots of $54 - 20\sqrt{2}$ are $2 - 5\sqrt{2}$ and $-2 + 5\sqrt{2}$. You may prefer to delay this until the solution of such pairs of equations is formally studied in Chapter 14.

Pages 167, 168

The purpose of the little section on irrational expressions is primarily to lay the groundwork for the subsequent study of irrational equations. Although the section on irrational expressions does to some extent foreshadow Chapter 9, we suggest that teachers do not introduce here, in any serious fashion, the words domain, range, or function. These words are systematically dealt with in Chapter 9 and more than a casual reference to them now will distract attention from irrational equations.

Page 172

Exercises 21 to 26 are included so that the teacher will have an opportunity to eliminate the notion, which is certain to have arisen by this time, that whenever an equation has a radical sign in it then both sides of the equation must be squared. This incorrect generalization arises because, in the equations thus far presented under the title "Irrational Equations" the variable has always appeared under a radical sign. This is a nice, though perhaps minor, illustration of one fundamental fact about learning a concept or generalization: contrast and variation are essential for the development of concepts and generalizations.

Chapter 9. Functions · Graphs · Variation

We think of this chapter as a transition chapter from the elementary study of structure and of sentence translation to the much more sophisticated investigation of sets of ordered pairs of numbers, the rules for constructing such sets, the manner of relating them to points of the plane, and their use in some of the applications of mathematics. It will also be useful as a reference chapter. Its ideas and vocabulary dominate the remainder of the book, and the student will notice frequent returns to the language and symbolism here presented.

The topics on pages 179-193 are the main content of the chapter, and they will be basic to the rest of the book. The sections on direct and inverse variation can be omitted entirely or taken up whenever the teacher deems it appropriate to do so. Sometimes

their introduction can be timed to suit the needs of the science courses. In any case, they are in the book because they provide useful language for discussing in a general way the very frequently occurring pattern, ab = c. If b is constant (or a) then a (or b) varies directly as c. If c is constant then a varies inversely as b. The language of variation is such a normal part of everyday talk that some study of it should probably be made in this course, although little use of it is made in subsequent mathematical discussions.

In this chapter is the first use of the word "function" in this book. There has simply been no need for the word prior to this chapter. To be sure, sets of ordered pairs of numbers have appeared before in an informal way in some graphing (see Ch. 3), but sets of ordered pairs have not presented themselves for serious study, and therefore no need existed for the distinction between functions and non-functions. From this point on in the book, however, there will be a great deal of study of functions, and the need for distinguishing between relations that are functions and relations that are not functions will become apparent.

We have used the ordered pair definition of a function because all the other variants in the definition of the word can be easily understood from this basis, and the set of ordered pairs is something the student can point to and say—this is a function. By using "f" for the function, "f(x)" for the value of the function at x, and "y = f(x)" as the defining equation, we find that directions as well as discussions can be written in unambiguous language, and the student finds the distinctions between f, f(x), and y = f(x) clear and easy to remember.

Page 193. Direct Variation

In connection with the function defined by an equation of the form $\frac{y}{x} = m$, where m is a non-zero constant, it helps to consolidate understanding of this relationship if the following line of talk occurs.

Individual elements of the function defined by $\frac{y}{x} = m$ may be singled out by the use of subscripts. Thus, $\frac{y}{x} = m$ is equivalent to $\frac{y_1}{x_1} = \frac{y_2}{x_2} = \frac{y_3}{x_3} = \ldots = m$, that is, the ordered pairs of f have in common the property that the ratio y : x is a constant.

Page 201

If the class has had a geometry course, then problems like Exercise 8 on page 201 provide an opportunity for some review of the geometry. Such a discussion might go along as follows. The problem is:

The area of an equilateral triangle varies as the square of the length of a side. When the side is 20 inches long, the area is 173 square inches to the nearest square inch. Find to the nearest tenth of a square inch the area of an equilateral triangle of side 4 inches.

The first sentence means the same as the equation $A = ks^2$. But you will remember that the area of an equilateral triangle is given by $A = \frac{e^2}{4}\sqrt{3}$. Hence the "k" should be equal to $\frac{\sqrt{3}}{4}$, or .433 to thousandths.

$$A = \frac{\sqrt{3}}{4}e^2$$

s	s^2	A
20	400	173
4	16	6.9

Depending on the class and the situation at the time, one could derive the formula $A = \frac{e^2}{4}\sqrt{3}$.

Thus, if $\triangle ABC$ is equilateral, with each side $\underline{e}$ units long and $CT \perp AB$,

then $AT = \frac{1}{2}e$ and $CT = \sqrt{e^2 - \frac{1}{4}e^2} = \frac{e}{2}\sqrt{3}$.

$$\therefore A = \frac{1}{2}\overline{AB} \cdot \overline{CT} = \frac{1}{2} \cdot e \cdot \frac{e}{2}\sqrt{3} = \frac{e^2}{4}\sqrt{3}.$$

One could go on and remind the students that Hero's formula for the area of a triangle, $A = \sqrt{s(s-a)(s-b)(s-c)}$, reduces to $\frac{e^2}{4}\sqrt{3}$ if $a = b = c$ and $s = \frac{a+b+c}{2}$ or $\frac{3e}{2}$.

Substituting in $A = \sqrt{s(s-a)(s-b)(s-c)}$

we have $A = \sqrt{\frac{3e}{2}(\frac{e}{2})(\frac{e}{2})(\frac{e}{2})}$

and $A = \sqrt{3(\frac{e}{2})^4}$

and $A = \frac{e^2}{4}\sqrt{3}$

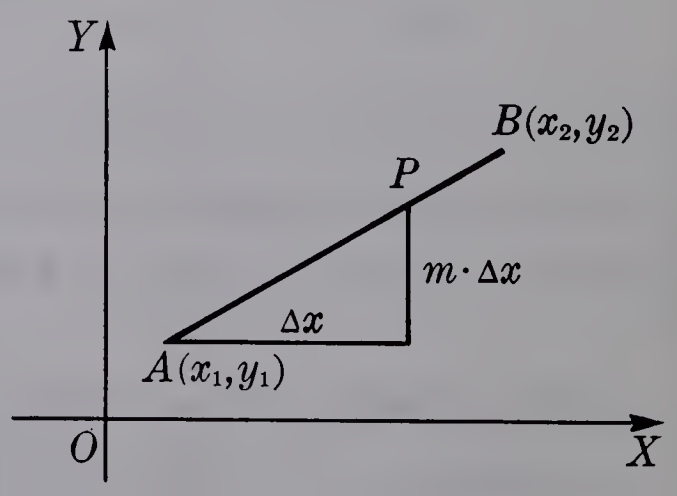
Page 207

While new discoveries in science and new developments in such fields as medicine and technology have long been newsworthy, reports of developments in mathematics have been rare in the public press. This is partly because there is little general understanding of mathematical concepts, and also because the need for special symbols makes the process of communicating mathematical ideas very difficult.

The fact, however, that mathematics is among the fastest-growing of all the sciences and that it is being more and more widely used in business and industrial operations is gradually forcing it into public notice. More frequently we are seeing in metropolitan newspapers reports such as the following. "The research session of the Eastern Annual Conference of the American Association of Advertising Agencies last week was given the basic principles of the linear programming process for selecting advertising media."

Industrial problems for which the methods of linear programming are used are generally so complex that their solution requires advanced mathematical procedures and the help of high-speed computers. The basic idea of linear programming is, however, a simple one. It can be understood by students who are familiar with the relationships between the coordinates of points on a line. In essence, this basic idea is that if the ordered pairs which are the coordinates of points on a given line segment AB are used as the replacement set for values of x and y in any linear expression $ax + by + c$, then the coordinates of the end-points of AB give the maximum and minimum values of the expression.

To see why this is so, suppose that the slope of the segment AB is m, and let A have coordinates (x_1, y_1), B have coordinates (x_2, y_2) with $x_2 > x_1$. Consider a point P on AB with abscissa $x_1 + \triangle x$ $(\triangle x > 0)$. The ordinate of P is $y_1 + m \cdot \triangle x$. The value of the expression $ax + by + c$ at P is $a(x_1 + \triangle x) + b(y_1 + m\triangle x) + c$ or $ax_1 + by_1 + c\triangle x(a + bm)$. Now $ax_1 + by_1 + c$ is the value of the expression at A, so we have value at P = value at A + $\triangle x(a + bm)$. For a given line segment



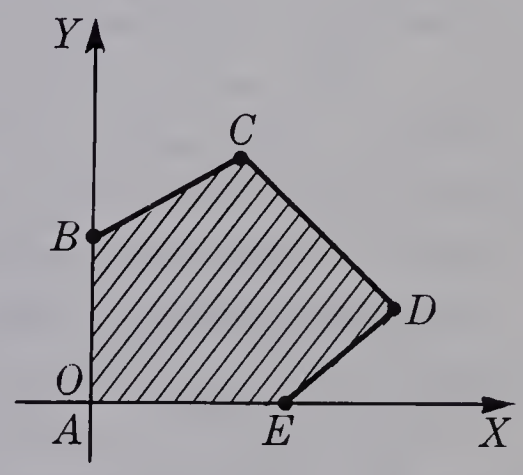
AB and a given expression ax + by + c, the values of m, a, b are fixed and so a + bm is a constant.

Hence as P takes positions on AB, one and only one of the following situations holds:

1. a + bm > 0. In this case, the value of ax + by + c at P exceeds the value at A by an amount proportional to $\triangle x$. The value of the expression is therefore greatest when $\triangle x$ is greatest, which is at B; and is least when $\triangle x = 0$, which is at A.
2. a + bm = 0. In this case, the value of ax + by + c is constant for all positions of P on AB.
3. a + bm < 0. In this case, the value of ax + by + c at P is less than the value at A by an amount proportional to $\triangle x$. The value of the expression is therefore greatest at A, and least at B.

Hence, in those cases where the value of the expression ax + by + c is not constant for points on AB, the maximum value occurs at one end of AB and the minimum value occurs at the other end.

A consequence of this result is that when values of ax + by + c are being considered for points of a convex polygonal set such as that in the shaded region, the maximum and the minimum values of the expression must occur at corner points. To find the maximum (or the minimum) value of the expression, therefore, it is only necessary to evaluate the expression at A, B, C, D and E.



An elementary example using this procedure is discussed in Sidelight 8, pages 207-209, of A Second Course in Algebra with Trigonometry.

Two articles on linear programming are contained in The Mathematics Teacher, March, 1960.

Chapter 10. Exponents and Logarithms

The study of exponents and logarithms at this time brings up again the ever-present dilemma that faces the teacher of algebra: it is desirable to reduce as much as possible the time spent on such mechanics, but if they are to be studied at all, enough time must be spent on them so that a useful level of skill is attained. We consider it quite important that the student be skillful in dealing with numbers that are expressed in a form having rational exponents. Subsequent work in mathematics demands easy familiarity with such forms. They appear particularly in differentiation, in applications involving very large or very small numbers, and in the study of the logarithmic function by the methods of the calculus. The actual arithmetic work of using a logarithm table may be short-cut as much as the teacher thinks desirable, but here again there is no point in doing a superficial job. Either the pupil can use the table efficiently or he cannot. So if you want him to use it at all, then drill enough to make his use of it efficient.

Page 213

The sets of problems on this page and on page 219 lend themselves well to oral drill except for a few at the bottom of page 219. They are, normally, worth doing two or three times. For example, one can do Exercises (A-1) on page 213 orally and assign Exercises (A-2) for part of the homework. Another day, do Exs. (A-2) in class and assign Exs. (A-1) for homework.

In connection with problems like $12^2 \div 3^2$, point out explicitly that the relation $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ is symmetric; that is, if $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, then $\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$. Hence $\frac{12^2}{3^2} = \left(\frac{12}{3}\right)^2 = 4^2 = 16$. Sometimes the problem and its solution are worth paraphrasing into words: To divide two powers that have the same exponent, but may or may not have the same bases, we divide the bases and raise the quotient to the common power.

The word "power" is commonly used for two different meanings. We speak of "2^4" as "2 to the fourth power." We also say, in a problem like $2^3 \times 2^5$, "to multiply two <u>powers</u> that have the same base, write the same base to the sum of the exponents." Thus, in 2^3 we may use the word "power" to refer to the 3 or to the entire

symbol 2^3. The context always makes clear what meaning is expected, but it is useful for the pupil to know that the double use exists.

Page 219

Encourage a certain amount of flexibility in doing some of these problems. Thus, to simplify $\left(\frac{1}{2}\right)^{-3}$ one may think "write the reciprocal of $\frac{1}{2}$ and then cube the reciprocal." Or, "$(1)^{-3} = 1$ and $2^{-3} = \frac{1}{8}$, therefore $\left(\frac{1}{2}\right)^{-3} = \frac{1}{\frac{1}{8}} = 8$."

If asked to simplify $\left(x^{-3}\right)^{-\frac{1}{3}}$, we may think:

$(-3)\left(-\frac{1}{3}\right) = +1$. Hence, the answer is x^1.

or $\left(x^{-3}\right)^{-\frac{1}{3}} = \frac{1}{\left(x^{-3}\right)^{\frac{1}{3}}} = \frac{1}{\left(\frac{1}{x^3}\right)^{\frac{1}{3}}} = \frac{1}{\left(\left(\frac{1}{x}\right)^3\right)^{\frac{1}{3}}} = \frac{1}{\frac{1}{x}} = x.$

or $\left(x^{-3}\right)^{-\frac{1}{3}} = \left(\frac{1}{x^3}\right)^{-\frac{1}{3}} = \frac{1}{\left(x^3\right)^{-\frac{1}{3}}} = \frac{1}{x^{-1}} = x.$

Students should not be permitted to carry the variations too far.

In a problem like $\frac{2x^{-2} - x^{-1}}{2x^{-2}}$, one may write $\frac{\frac{2}{x^2} - \frac{1}{x}}{\frac{2}{x^2}} = \frac{2 - x}{2}$, or write $\frac{2x^{-2} - x^{-1}}{2x^{-2}} \cdot \frac{x^2}{x^2} = \frac{2 - x}{2}$.

Pages 225 ff.

By the time the class arrives at this page, so much work has been done with exponents in general and with powers of 10 in particular that an elegant motivation for the study of logarithms can be set up by an approach such as the following.

Consider the problem of carrying out the sequence of operations in the exercise

$$\frac{\sqrt[3]{536.8} \times (40.81)^4}{0.0003156}.$$

Patience, a large table of cube roots, and much care would enable one to obtain an answer to such a problem. However, our knowledge of exponents now gives us a much clearer approach to the problem. (The main steps in the argument are marked A and B)

A. We write each numeral as some power of ten:

$$\frac{\sqrt[3]{10^x} \times (10^y)^4}{10^z}.$$

B. Then we use the laws of exponents:

$$\frac{10^{\frac{x}{3}} \times 10^{4y}}{10^{z}} = \frac{10^{\frac{x}{3}+4y}}{10^{z}} = 10^{\frac{x}{3}+4y-z}.$$

The number represented by the expression $\frac{x}{3} + 4y - z$ is the power of 10 which gives the desired answer. Now x, y, and z represent numbers that are recorded in tables of logarithms. They are therefore available to us, and so the problem is solved by finding the value of $\frac{x}{3} + 4y - z$ and then reversing the use of the table. We think that an approach such as this is good because, if you make the original problem difficult enough, students will see the advantage of the procedure. In fact, they are usually quite delighted with it.

Pages 233-234

If the pupil has difficulty understanding this discussion of the negative characteristic, the following variation in language sometimes is helpful. Change the array shown on page 234 as follows:

$$\begin{aligned}
542.5 &= 5.425 \times 10^{2} = 10^{0.7344} \times 10^{2} = 10^{2.7344} \\
54.25 &= 5.425 \times 10^{1} = 10^{0.7344} \times 10^{1} = 10^{1.7344} \\
5.425 &= 5.425 \times 10^{0} = 10^{0.7344} \times 10^{0} = 10^{0.7344} \\
.5425 &= 5.425 \times 10^{-1} = 10^{0.7344} \times 10^{-1} = 10^{0.7344+(-1)} \\
&\qquad = 10^{-0.2656} \\
.05425 &= 5.425 \times 10^{-2} = 10^{0.7344} \times 10^{-2} = 10^{0.7344+(-2)} \\
&\qquad = 10^{-1.2656} \\
.005425 &= 5.425 \times 10^{-3} = 10^{0.7344} \times 10^{-3} = 10^{0.7344+(-3)} \\
&\qquad = 10^{-2.2656}
\end{aligned}$$

Etc.

Then the talk may be:

If we carried out the addition of the exponents in the last three lines then the digits, 2656, would have to be found elsewhere in the table than in the place where we found 7344. Such a situation would double the size of the tables we would need. Hence we use a form which indicates the addition without actually carrying it out:

$$\begin{aligned}
0.7344 + (-1) &= 9.7344-10 = \bar{1}.7344 \\
0.7344 + (-2) &= 8.7344-10 = \bar{2}.7344
\end{aligned}$$

Etc.

That is, the symbol $\overline{2}.7344$ is simply another name for the number that is represented by -1.2656. The symbol $\overline{2}.7344$ is a binomial. It is equivalent to (-2) + (+0.7344), etc.

Page 243

Sometimes the student has a little trouble with these exercises. When this happens, write out the solution in detail. Here is an illustration:

$$2 \log 4 - \tfrac{1}{3} \log 8$$

Let $2 \log 4 = x$

then $\log 4^2 = x$

$10^x = 16$ (A)

Let $\tfrac{1}{3} \log 8 = y$

$\log 8^{\frac{1}{3}} = y$

$10^y = 2$ (B)

Since we seek a value for x - y, we divide (A) by (B):

$10^{x-y} = 8$

$\therefore (x - y) \log 10 = \log 8$

Whence $x - y = \log 8$

Since we obviously don't want the student to take such a long route as a permanent procedure, encourage him, after the above sequence is understood, to think as follows:

$$2 \log 4 - \tfrac{1}{3} \log 8.$$

$$2 \log 4 = \log 4^2 = \log 16$$

$$\tfrac{1}{3} \log 8 = \log 8^{\frac{1}{3}} = \log 2$$

and so, $\log 16 - \log 2 = \log \frac{16}{2} = \log 8.$

Page 244

The principle underlying the solution of exponential equations is that if two positive numbers are equal, then their logarithms are equal. If a student raises a question about this statement, the following argument usually convinces him.

Theorem: If $a = b$ then $\log a = \log b$.

Let $x = \log a$

$y = \log b$

$$\text{Then } 10^x = a$$
$$10^y = b$$
$$\text{But } a = b$$
$$\therefore 10^x = 10^y$$
$$\text{and } x = y.$$

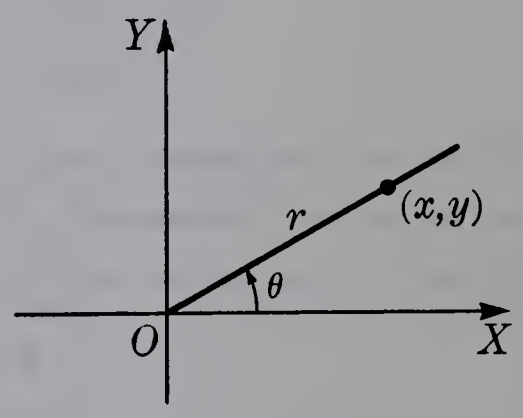
All this achieves, actually, is a paraphrase of the original statement, but it seems to satisfy. Also, it serves to emphasize the one-to-one correspondence between positive real numbers and their logarithms.

Chapter 11. Measurement in Right Triangles

The development of the trigonometric functions in the book proceeds in three stages. We first (page 249) define them by using the unit circle, restrict the domain (see page 250) to the set $0° < \theta < 90°$, and immediately connect the definition to the right triangle.

In Chapter 15 (pages 405 ff.), the sine and cosine are redefined so as to associate the x and the y with any point of the Cartesian plane. That is,

$\sin \theta = \frac{y}{r}$, $\cos \theta = \frac{x}{r}$, where $x^2 + y^2 = r^2$.



These definitions are in agreement with those given in this chapter, and allow us to operate with angles unrestricted in size. The concept of reference angle is developed. The Law of Sines and the Law of Cosines are developed, and, since work with geometric figures is still the primary consideration, the chapter concerns itself with sines and cosines of all angles.

Chapter 16 completes the definitions of the six functions for angles of any size. Each element of the domain is still thought of as the measure of an angle, as was the case in Chapters 11 and 15. All the uses to which the functions were put in these chapters were related to measurement in triangles.

In Chapter 19, however, the definitions are made in terms of sets of ordered pairs of real numbers without any direct association with the concept of an angle. This completes the theoretical development of the trigonometric functions and prepares the way for the discussions relevant to a calculus course. The teacher

is urged to read, at one sitting, the complete development. This would include the material on pages 249, 250, 405, 406, 408, 409, 429, 430, 532, 533, 534, 535, 536, 537. By doing this, a perspective on the development can be obtained, and this perspective will affect the manner of handling these functions in the classroom.

Chapter 12. Elements of Coordinate Geometry

The manner of using this chapter will vary considerably in accordance with the previous training of the class. For those who studied this subject matter in A Course in Geometry, Plane and Solid, the only thing new is pages 288-296 where related changes, the linear function, and functional notation are discussed. The chapter, up to page 288, would be primarily review for such a group. (The teacher will find that many numbers have been changed in the exercises.) The pupil already has, from Chapter 9, the basic vocabulary and the fundamental hypothesis that there is a one-to-one correspondence between pairs of numbers and points of the plane.

In general, we think the chapter breaks up rather naturally into two main parts. In the first part (pages 264-287) the basic motivation is the elaboration of the if-then pattern of thinking. Thus, if two points, $P(x_1, y_1)$ and $Q(x_2, y_2)$ determine a line, then we can deduce, in terms of the four numbers, x_1, x_2, y_1, y_2, such information as the length of PQ, the midpoint of PQ, the direction of PQ (slope), a criterion for parallelism and one for perpendicularity. On pages 277-279, some proofs of geometric theorems are illustrated, and others are called for. On pages 279-287, various aspects of the graph as a set of points are developed, with additional emphasis on the deduction of new information from given information. In this connection, the teacher should not forget that the assumption that "the graph of the set of ordered pairs defined by $ax + by + c = 0$ is a straight line" is made on page 187.

The second part of the chapter (pages 288-295) is easily motivated as the first serious investigation into related changes that the student has made. He is already acquainted with the notion of related changes, but has done little with it prior to this point in the course. The increased sophistication of the discussion presented here prepares the way for similar studies of

the quadratic and cubic functions, and lays a thorough foundation of vocabulary and concepts for the corresponding investigations in the calculus.

Page 279

We assume (on page 187) that the graph of an equation of the form $ax + by + c = 0$ is a straight line. Some curious and intelligent pupil might want a proof, and if he has had a geometry course then perhaps his want should be satisfied. We did not include the proof in the text because it would have required more space and time than we thought appropriate. However, the teacher should have available a proof. Here is one based on the assumption that the slope of the segment connecting any two points on a line is a constant.

Theorem. The locus of points whose coordinates are related by the equation $ax + by + c = 0$ is a straight line.

As with any locus theorem, we must prove two theorems, one the converse of the other. These theorems may be conveniently formulated if we make certain preliminary statements:

Let $P\ (x_1, y_1)$ and $Q\ (x_2, y_2)$ be two ordered pairs of numbers that satisfy $ax + by + c = 0$. Let ℓ be the line determined by the points P and Q. The point M has coordinates (x_0, y_0). Then the two theorems are:

A. If M is on ℓ, then (x_0, y_0) satisfy $ax + by + c = 0$.

B. If (x_0, y_0) satisfy $ax + by + c = 0$, then M is on ℓ.

Proof of A:

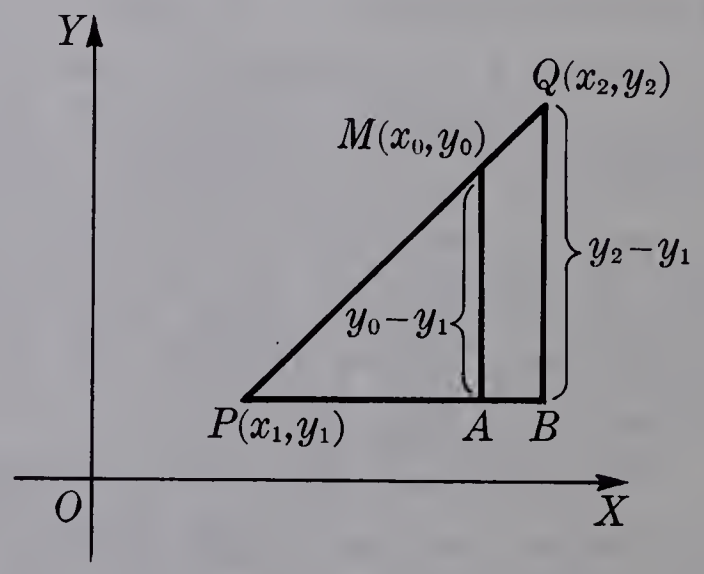
Since M is on ℓ, then

$$\triangle PMA \sim \triangle PQB$$

and so
$$\frac{MA}{PA} = \frac{QB}{PB}$$

or
$$\frac{y_0 - y_1}{x_0 - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

Now
$$\begin{cases} ax_1 + by_1 + c = 0 \quad (1) \\ ax_2 + by_2 + c = 0 \end{cases}$$



Then $a(x_2 - x_1) + b(y_2 - y_1) = 0$

or
$$\frac{y_2 - y_1}{x_2 - y_2} = \frac{-a}{b}$$

$$\therefore \quad \frac{y_0 - y_1}{x_0 - x_1} = \frac{-a}{b}$$

$$by_0 - by_1 = -ax_0 + ax_1$$

$$ax_0 + by_0 + (-ax_1 - by_1) = 0$$

But $\quad -ax_1 - by_1 = c \quad$ from (1)

$$\therefore \quad ax_0 + by_0 + c = 0$$

Proof of B

Since (x_0, y_0) satisfies

$$ax + by + c = 0,$$

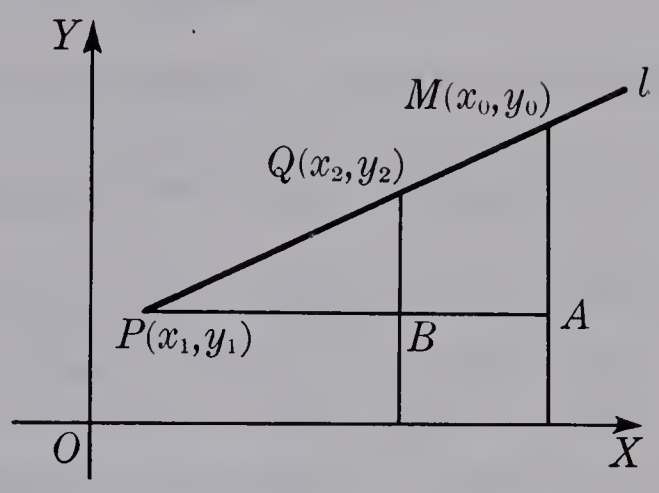
then $\quad ax_0 + by_0 + c = 0.$

But $\quad ax_1 + by_1 + c = 0.$

$$\therefore \quad a(x_0 - x_1) + b(y_0 - y_1) = 0$$

$$\frac{y_0 - y_1}{x_0 - x_1} = \frac{-a}{b}$$

$$= \text{slope of PM}$$



Now $\frac{-a}{b}$ is the slope of $ax + by + c = 0$, that is, the slope of ℓ. Hence PQM is a straight line since PQ and PM have equal slopes. M lies on ℓ.

Pages 280-283

We refer to "The Point-Slope Equation of a Line." In the section above we examine, without labelling it, the slope-intercept form of the equation of a line. The problem of writing the equation of a line may profitably be put into more general form, however, in the following way.

It is characteristic of a straight line that the slope of any portion of it equals the slope of any other portion of it. Hence we will always have the equation of a line when we write this property in symbolic form, using (x, y) to represent the coordinates of <u>any</u> point on the line. Another way to say the same thing is that since a line is determined by two independent conditions we should have the equation of a line whenever we combine the two given expressions for the slope into a single sentence. The latter always asserts that the two expressions are equal because they each stand for the slope of the same line. To illustrate, we list below several pairs of conditions which determine a line and give

the equation in the form first obtained by thinking of the equation as a statement of equality of two expressions for the slope.

(1) two points: (2, 3); (-1, 5)	(1) $\frac{y - 3}{x - 2} = \frac{5 - 3}{-1 - 2}$
(2) point-slope: (2, 3), m = -3	(2) $\frac{y - 3}{x - 2} = -3$
(3) Slope, y-intercept: m = 4, y-intercept = 7	(3) $\frac{y - 7}{x - 0} = 4$
(4) Through (5, 4) parallel to x - y = 1	(4) $\frac{y - 4}{x - 5} = 1$
(5) Through (5, 4) perpendicular to 2x - y = 1	(5) $\frac{y - 4}{x - 5} = \frac{1}{2}$

Notice in examples (4) and (5) that it is certainly handy to know that when the equation is in the form $y = mx + k$ then m is the slope. This little fact is so useful that the student should be encouraged to remember it—at least for the duration of this course! But it is also worthwhile to point out that when an equation of a line is given, e.g., $x - y = 1$, then the slope can be found by using the definition of slope.

Thus, if $x - y = 1$
then when $x = 2,\ y = 1$
and when $x = 3,\ y = 2.$
And so, $\frac{\triangle y}{\triangle x} = \frac{2 - 1}{3 - 2} = 1$

Keep on going back to definitions whenever it can be done without making the solution of the current problem unreasonably cumbersome. By doing so, we reach a better approximation to a course that is suitable for general education.

Page 287—Exercise 1

Here occurs the first use in the book of the word "locus." Since it occurs in a "B" exercise, we have not stopped at this point to discuss the word. If there is the slightest hesitancy by the pupil in accepting the word (most of them will have used it in a geometry course), then tie it up promptly with the discussion of graphs on pages 185 to 187. A locus is a set of points every one of which satisfies a specified condition, and every point which satisfies the condition is a member of the set. If the class has had a formal course in geometry, then associate the word "locus" with the various common geometric loci: angle-

bisector, perpendicular-bisector of a line segment, etc. The word locus occurs casually again on page 319, and it appears formally on page 321 in the definition of the parabola. It occurs again in problems on page 326 and is extensively used in Chapter 14. It is essentially synonymous with the concepts of "graph" and "set of points." A non-technical, informal sentence which sometimes proves helpful is: "A locus is the place where you will find all those points, and no others, which satisfy a given condition."

It is possible to get along quite comfortably without the word. We introduce it here simply because it is convenient and undoubtedly familiar to most of the students. Directions beginning "Write the equation of the locus . . ." can be rephrased to "Write the equation which defines the set of points . . ."

Chapter 13. The Quadratic Function

The investigation of the quadratic function presented in this chapter is fairly complete and detailed. The teacher should have the broad outlines of the development in mind so that the presentations and discussions in class have coherence and motivation. The sequence goes like this:

A. The basic facts about the quadratic function: domain, range, and zeros.

B. The graph of a specific quadratic function.

C. Locus definition of parabola.

D. Association of quadratic function with parabola; vertex of parabola, graph of $y = ax^2 + bx + c$; range of y.

E. Graphical solution of quadratic equations and quadratic inequalities; complex numbers, equality of, operations with, order of; the nature of the roots of $ax^2 + bx + c = 0$, sum and product of the roots.

F. The formation of quadratic functions and equations.

Page 319

The remark that the graphs of $y = 4x - x^2$ and $y = x^2 - x - 6$ are identical except for their location with respect to the axes may lead some inquisitive student to ask "How can you be sure that they are identical?" There are three explanations that may be used. Pick the one that you think your questioner is most ready to understand. We complete the square in each case:

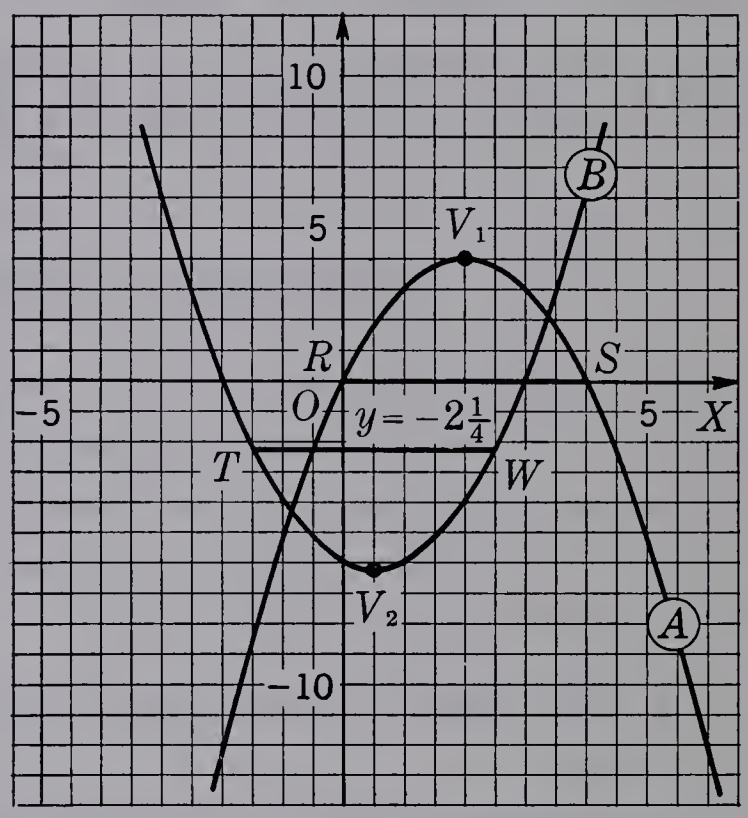
1. $y = 4x - x^2 = -(x - 2)^2 + 4$
$y = x^2 - x - 6 = \left(x - \frac{1}{2}\right)^2 - 6\frac{1}{4}$

Then the discussion goes like this — The only effect of the negative coefficient of x^2 is to revolve the curve through 180° around the x-axis pivoting on the x-intercepts. The "4" and the "$6\frac{1}{4}$" merely move the curve up or down along the y-axis. The "2" and the "$\frac{1}{2}$" move the curve to the left. In other words, the basic curve is $y = ax^2$, and the only constant that affects the shape of the curve is a. The trouble with this answer is that the translation left or right takes a little time for the student to understand.

2. $y = -(x - 2)^2 + 4$ (A)
$y = \left(x - \frac{1}{2}\right)^2 - 6\frac{1}{4}$ (B)

The vertex of (A) is the point V_1: (2, 4)
The vertex of (B) is the point V_2: $\left(\frac{1}{2}, -6\frac{1}{4}\right)$

If the curves are identical then the length of a chord of (A) that is parallel to the x-axis and K units from V_1 should be the same as the length of a chord of (B) that is parallel to the x-axis and K units from V_2.



In (A), the line $y = 0$ is 4 units from V_1. It cuts the curve in R: (0, 0) and S: (4, 0). Hence RS is 4 units long.
In (B), the line $y = -2\frac{1}{4}$ is 4 units from V_2. In $y = x^2 - x - 6$, we put $y = -2\frac{1}{4}$, getting $4x^2 - 4x - 15 = 0$.

Whence $x = \frac{5}{2}, -\frac{3}{2}$

Hence the line $y = -2\frac{1}{4}$ cuts (B) in T: $\left(-\frac{3}{2}, -2\frac{1}{4}\right)$ and W: $\left(\frac{5}{2}, -2\frac{1}{4}\right)$. Hence $TW = \left|-\frac{3}{2} - \frac{5}{2}\right| = 4$. So $RS = 4 = TW$.

3. If we reflect $y = 4x - x^2$ in the x-axis and translate the curve so that its vertex is at V_2, we should end up with an equation that is identical with $y = x^2 - x - 6$.

When reflected, we have

$$y = x^2 - 4x \qquad \text{(C)}$$

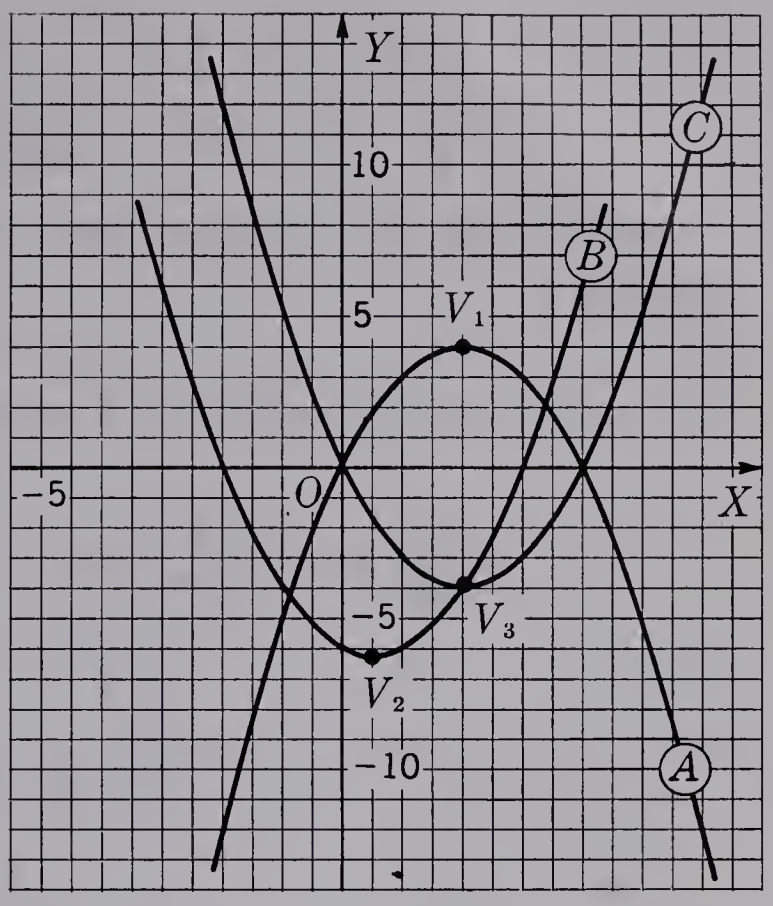
The vertex of (C) is V_3: (2, -4)

We want this vertex to be $\left(\frac{1}{2}, -6\frac{1}{4}\right)$

$$y + 2\frac{1}{4} = \left(x + \frac{3}{2}\right)^2 - 4\left(x + \frac{3}{2}\right)$$

$$y + 2\frac{1}{4} = x^2 + 3x + \frac{9}{4} - 4x - 6$$

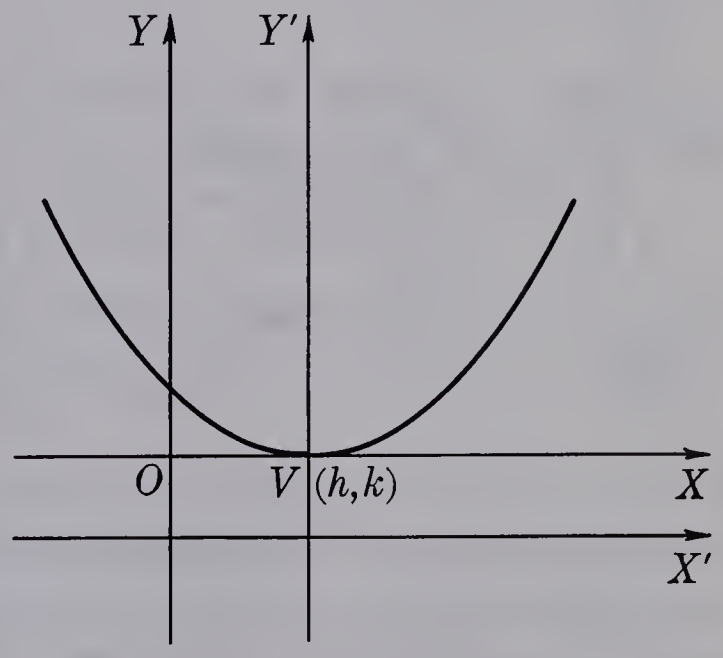
$$y = x^2 - x - 6$$



Pages 321-323

"It can be shown that the graph of a quadratic function is in every case a parabola."

First, recall that the coordinate axes may be moved in any manner that serves a useful purpose. Usually the purpose is to achieve a simplification of the equation which defines a curve. For example, if the graph of $y = ax^2 + bx + c$ is as shown in the diagram, we may move the y-axis to the right parallel to itself to the position indicated by Y', and the x-axis may be moved parallel to itself to the position indicated by X'. Then, the equation of the curve, referred to the new axes, is of the form $y' = a(x')^2$. The coordinates of V with respect to the old axes are (h, k). With respect to the new axes the coordinates are (0, 0). Thus $x' = x - h$ and $y' = y - k$, and these equations are general equations which describe the effect of translating the y-axis through h units and the x-axis through k units. The movement of y is to the right or left and the movement of x is up or down, depending on the algebraic signs of h and k.



To transform the equation $y = ax^2 + bx + c$ into the form $y' = a(x')^2$ we make the substitutions

$$x = x' + h$$
$$y = y' + k$$

So that $y' + k = a(x' + h)^2 + b(x' + h) + c$

$$y' + k = a(x')^2 + (2ah + b)x' + ah^2 + bh + c$$

$$y' = a(x')^2 + (2ah + b)x' + ah^2 + bh + c - k$$

Since the x' term and the constant term are each to be zero, we have

$$\begin{cases} 2ah + b = 0 \\ ah^2 + bh + c - k = 0 \end{cases}$$

$$h = -\frac{b}{2a}$$

$$a\left(\frac{b^2}{4a^2}\right) + b\left(-\frac{b}{2a}\right) + c - k = 0$$

Whence

$$\begin{cases} k = c - \frac{b^2}{4a} \\ h = -\frac{b}{2a} \end{cases}$$

(A) $$\therefore y' = a(x')^2$$

Now any conclusion not dependent on the choice of axes that we can make about $y' = a(x')^2$ will be a valid conclusion about $y = ax^2 + bx + c$.

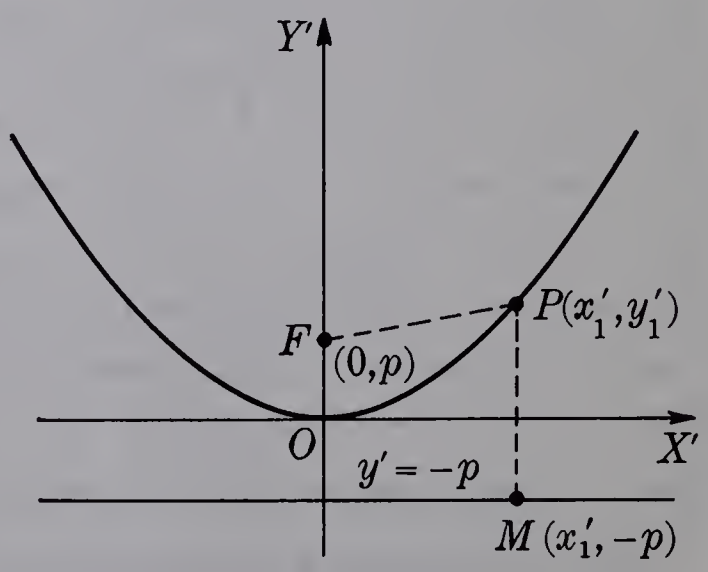
Thus, consider any point P: (x'_1, y'_1) which satisfies the definition of a parabola, i.e., that PF = PM. We have

$$\sqrt{(x'_1 - 0)^2 + (y'_1 - p)^2} = \sqrt{(x'_1 - x'_1)^2 + (y'_1 + p)^2}$$

$$(x'_1)^2 + (y'_1)^2 - 2py'_1 + p^2 = (y'_1)^2 + 2py'_1 + p^2$$

(B) $$(x'_1)^2 = 4py'_1$$

Hence, any point which satisfies the definition of a parabola has coordinates which satisfy an equation of the form $y' = a(x')^2$. The coordinates of any such point also satisfy an equation of the form $y = ax^2 + bx + c$.



When we have the form (B) we see that the focus of the parabola is the point (0, p) and the directrix is the line $y' = -p$. Thus, from (B) and (A) we have $a = \frac{1}{4p}$, whence $p = \frac{1}{4a}$.

If, now, we refer p back to the original axes, then

$$p = \frac{1}{4a} = y' = y - k = y - \left(c - \frac{b^2}{4a}\right)$$

Whence, $y = \frac{1}{4a} + c - \frac{b^2}{4a}$

$x = x' + h = 0 + \left(-\frac{b}{2a}\right) = -\frac{b}{2a}$

Hence the coordinates of the focus, referred to the original axes, are $\left(-\frac{b}{2a}, \frac{4ac - b^2 + 1}{4a}\right)$.

The equation of the directrix, referred to the original axes, is obtained as follows:

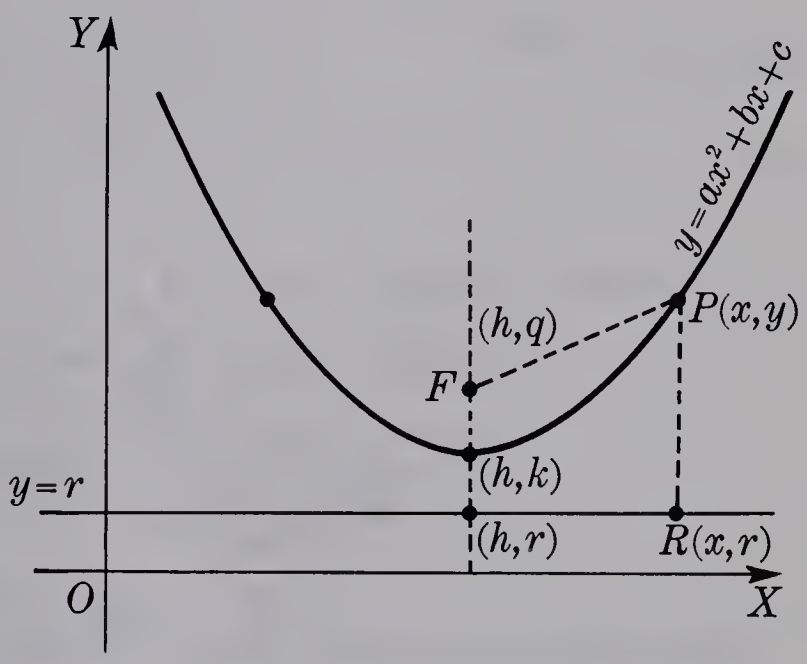
$y' = -p = -\frac{1}{4a}$ (from Ⓐ and Ⓑ)

$y = y' + k = -\frac{1}{4a} + c - \frac{b^2}{4a}$

$\therefore\ y = \frac{4ac - b^2 - 1}{4a}$ is the equation of the directrix, referred to the original axes.

These formulas for the focus and the directrix of $y = ax^2 + bx + c$ can be arrived at without a change of axes. The argument follows.

$$\begin{aligned} y &= ax^2 + bx + c \\ &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2}\right) \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \end{aligned}$$



The equation of the axis of symmetry of any parabola can be found by taking the average of the zeros $\left(x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$. $\frac{\frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}}{2} = \frac{-2b}{4a} = \frac{-b}{2a}$. Therefore, the equation of the axis of symmetry is $x = -\frac{b}{2a}$.

Substituting this x-value in the equation $y = ax^2 + bx + c$, we have: $y = a\left(\frac{b^2}{4a^2}\right) + b\left(\frac{-b}{2a}\right) + c = \frac{b^2}{4a} - \frac{b^2}{2a} + c = \frac{-b^2}{4a} + c$.

In the diagram, $h = -\frac{b}{2a}$ and $k = c - \frac{b^2}{4a}$. By definition of the parabola, $q - k = k - r$, and $PF = PR$.

Since $PF = PR$, we can use the distance formula:

$$\sqrt{(x-h)^2+(y-q)^2} = \sqrt{(x-x)^2+(y-r)^2}$$

$$x^2 - 2hx + h^2 + y^2 - 2qy + q^2 = y^2 - 2ry + r^2$$

$$y(2r-2q) = -x^2 + 2hx - h^2 - q^2 + r^2$$

$$y = \left(\frac{-1}{2r-2q}\right)x^2 + \left(\frac{h}{r-q}\right)x - \frac{h^2}{2r-2q} + \frac{r+q}{2}$$

Comparing these coefficients with the general quadratic, $y = ax^2 + bx + c$:

$$a = \frac{-1}{2r-2q} \qquad 2a(r-q) = -1 \qquad (1)$$

$$b = \frac{h}{r-q} \qquad b(r-q) = -\frac{b}{2a} \qquad (2)$$

$$c = \frac{r+q}{2} - \frac{h^2}{2(r-q)} \qquad 2c(r-q) = (r^2 - q^2 - h^2) = r^2 - q^2 - \frac{b^2}{4a^2} \qquad (3)$$

From the diagram: $q = 2k - r = 2c - \frac{b^2}{2a} - r$ (4)

Therefore, $r - q = r - (2c - \frac{b^2}{2a} - r) = 2r - 2c + \frac{b^2}{2a}$ (5)

and $r + q = r + (2c - \frac{b^2}{2a} - r) = 2c - \frac{b^2}{2a}$ (6)

In (1) above, $2a(r-q) = -1$

$$2a\left(2r - 2c + \frac{b^2}{2a}\right) = -1$$

$$4ar - 4ac + b^2 = -1$$

$$r = \frac{4ac - b^2 - 1}{4a}$$

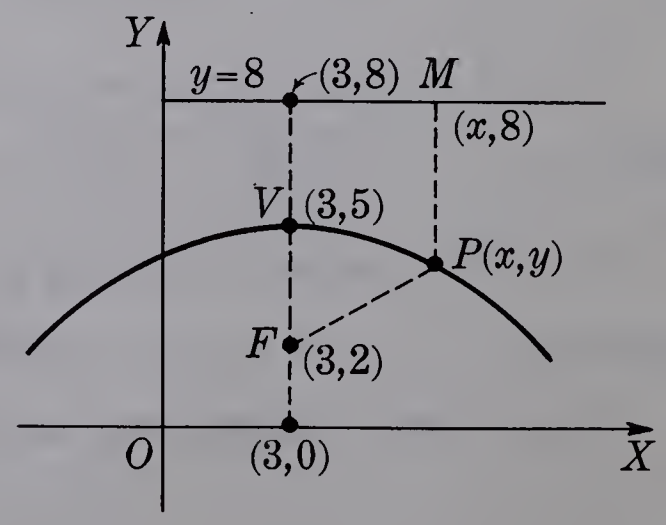
In (4) above, $q = 2c - \frac{b^2}{2a} - \frac{4ac - b^2 - 1}{4a}$

$$= \frac{8ac - 2b^2 - 4ac + b^2 + 1}{4a} = \frac{4ac - b^2 + 1}{4a}$$

The focus (h, q) is $\left(-\frac{b}{2a}, \frac{4ac - b^2 + 1}{4a}\right)$

The directrix is the line $y = r$, or $y = \frac{4ac - b^2 - 1}{4a}$

As a check on these derivations, the student will be interested in a procedure such as that outlined below where we choose a focus and a directrix, derive the equation of the parabola having this focus and directrix, and then use the formulas above to check the focus and directrix.



In the parabola with vertex

(3, 5), if we choose a focus with coordinates (3, 2), then the directrix is the line $y = 8$. Using the point (x, y), the equation defining the parabola can be derived thus:

$(x - 3)^2 + (y - 2)^2 = (x - x)^2 + (y - 8)^2$, and

$y = -\frac{1}{12}x^2 + \frac{1}{2}x + \frac{17}{4}$.

Then, using $a = -\frac{1}{12}$, $b = \frac{1}{2}$, and $c = \frac{17}{4}$, in the formulas for the coordinates of the focus, $x = -\frac{b}{2a}$ and $y = \frac{4ac - b^2 + 1}{4a}$, we have: $-\frac{b}{2a} = -\frac{\frac{1}{2}}{-\frac{1}{6}} = 3.$

$$\frac{4ac - b^2 + 1}{4a} = \frac{4(-\frac{1}{12})(\frac{17}{4}) - \frac{1}{4} + 1}{4(-\frac{1}{12})} = \frac{-\frac{17}{12} + \frac{3}{4}}{-\frac{1}{3}} = \frac{-17 + 9}{-4} = 2.$$

And, in the formula for the directrix, $y = \frac{4ac - b^2 - 1}{4a}$, we have: $y = \frac{4(-\frac{1}{12})(\frac{17}{4}) - \frac{1}{4} - 1}{4(-\frac{1}{12})} = \frac{-\frac{17}{12} - \frac{5}{4}}{-\frac{1}{3}} = \frac{-17 - 15}{-4} = 8.$

Page 326 — Problem 5

"Find the equation of the locus of points which are the same distance from (0, 1) as from the x-axis."

We suggest explanatory language for this problem of the following sort:

"If we denote the coordinates of any point of this set of points by the symbol, (x, y), then the condition imposed upon (x, y) by the statement of the problem is $\sqrt{(x - 0)^2 + (y - 1)^2} = y$," etc. The importance of the quantifier "any" should be stressed.

Page 328. Range of a Quadratic Function

In the study of the linear function, the question of range was trivial. In the study of the general cubic, coming up later in Chapter 17, the question is also trivial. For the quadratic, however, and for the other conic sections, the problem is not trivial. With a moderately alert class it will be worthwhile to mention the problem of range in the study of such functions as $y = \sqrt{x^2 - 4}$, $y = \sqrt{4 - x^2}$. Sometimes, also, a sensible opening occurs at this point for a reference to relative maxima and relative minima as exemplified by the cubic function. For a discussion of relative maxima and minima, see page 82 of this manual.

Page 334—Exercises (A)

These graphs are not without interest and value at this stage, but it is doubtful if there is point in doing more than a couple of them. The type shown in Exercise 4 is often of special interest because it illustrates the simplicity of using a template for the curve $y = x^2$, and then merely plotting straight lines.

Inequalities

Some attention to inequalities at this time helps to sharpen the pupil's understanding of the equalities and rounds out the study of $ax^2 + bx + c$. We suggest that the work on pages 20-24 be reviewed, and that a consolidation of the pupil's understanding of inequalities be striven for.

The teacher may wish to introduce the word "interval" at this time. We have not done so in the text simply because there is no real need for it and, in our opinion, it therefore becomes an unnecessary expansion of vocabulary. Since an interval is simply a set of numbers defined by one or more inequalities, it seemed pointless to burden the pupil with another new word. An "open interval" determined by two numbers a and b, where $a < b$, is the set of all numbers x for which $a < x < b$. The "closed interval" determined by the same a and b is the set of all numbers x for which $a \leqq x \leqq b$.

Page 336—Complex Numbers

An ordinary college capable class can read pp. 336-342 as an assignment and do, for example, the odd numbers on page 343 as part of the same assignment. In fact, this is a good spot to give the class some practice in reading the text before you have talked about it. Then, after they have read the text and done the problems on page 343 or the problems on pages 339, 340, 341, the teacher should summarize the discussion of the text, point out the aspects that he thinks should be emphasized, and then go on. Two days and two assignments are quite enough for pp. 336-345. Of course, if your class is mature and eager, you may wish to discuss at this time the formal development of the complex numbers as given in Sidelight 10 at the end of this chapter. We find that this development is nice to have available for the occasional pupil who has exceptional interest and skill in the more formal aspects of the subject. Rarely will you find a whole class of

eleventh graders who should be presented with this much formality. If the pupils read through pages 336-342 for one assignment, they will have a coherent and accurate comprehension of the role of complex numbers at this stage of their study, and this is all that most of them can deal with.

Page 340—Division

Somewhere in your oral observations on complex numbers, emphasize the approach to division used here: that, by definition, division of w by 2 is equivalent to multiplication of w by $\frac{1}{2}$. This ties the process to previously acquired knowledge of the structure of the subject, and reminds the student once more of its coherence and unity.

Page 341

Fourth line from the bottom of the page: "For most purposes the form $i\sqrt{a}$ in (2) is preferable to $\sqrt{-a}$." We think it so important for the student to form the habit of always changing from the form $\sqrt{-a}$ to $i\sqrt{a}$ before performing the simplest operations that we urge the teacher to insist on this procedure at all times.

Pages 348-353. Sum and Product of the Roots of $ax^2 + bx + c = 0$

Entertaining puzzle problems can be developed using these relationships between the roots and the coefficients of the equation. They are helpful in forming equations when the roots are given, especially when the roots are conjugate complex numbers.

These relationships are useful in expediting the procedure for finding the extreme value of a quadratic function. Thus, if the zeros of $y = ax^2 + bx + c = f(x)$ are x_1 and x_2, then we have seen that the axis of symmetry of $y = f(x)$ is $x_m = \frac{1}{2}(x_1 + x_2)$ and so the extreme value of the function is $f(x_m)$. (See p. 328 of the algebra text.) But if x_1 and x_2 are the zeros of $y = f(x)$, then x_1 and x_2 are the roots of $f(x) = 0$, their sum is $-\frac{b}{a}$ and hence $-\frac{b}{2a} = x$ is the axis of symmetry and the extreme value of $f(x)$ is $f\left(-\frac{b}{2a}\right)$.

The fact that $-\frac{b}{a}$ is the sum of the roots of $ax^2 + bx + c = 0$ can also be used in the problem of finding the midpoint of a chord.

Example: Find the midpoint of the chord of the parabola defined by the equation $y = -\frac{2}{3}x^2 + \frac{15}{4}x - \frac{1}{12}$, if the chord has the equation $x + 4y - 13 = 0$.

Solution: Since $y = \frac{13 - x}{4}$, we have $\frac{13 - x}{4} = -\frac{2}{3}x^2 + \frac{15}{4}x - \frac{1}{12}$.

Whence, $x^2 - 6x + 5 = 0$.

Hence, the x-coordinates of the endpoints of the chord have a sum of 6, and the x-coordinate of the midpoint of the chord is therefore 3. The y-coordinate is $y = \frac{13 - x}{4} = \frac{13 - 3}{4} = \frac{5}{2}$.

Therefore, the midpoint of the chord is $\left(3, \frac{5}{2}\right)$.

Chapter 14. Equations of the Second Degree and Their Graphs

Recall the note of page 54 of this manual concerning Exercise 1 on page 287 of the text. We repeat here the statement that it is really quite unimportant to use the word locus. "Set of points" is a synonym for it and often produces a clearer concept. We just think it bad pedagogy to ignore the word if the class has previously made contact with it.

Many objectives could be written down for this chapter. The basic one is to effect a consolidation of the idea that a set of ordered pairs of numbers that are related in a specified way may be placed in a one-to-one correspondence with points of the plane, and when this is done the points will be in a pattern that we identify by a name. When we have two sets of points, then the intersection of these sets consists of the set of points common to both of them. The arithmetic analogue is the set of ordered pairs of numbers which make up the solution set of the system of equations. Hence the chapter carries along concurrently the graphing of sets of points and the algebraic solution of systems of equations.

The work of this chapter is difficult to handle properly, for there is a built-in dichotomy between the big ideas that the teacher wants to keep before the student and the technical details that now become difficult enough to require considerable time and effort. Consequently, the teacher must always be aware of the danger of becoming bogged down in the details and forgetting the big ideas, and also of the danger of talking about the big ideas so much that insufficient time is left for the development of skills. Mere awareness by the teacher will do much to minimize this danger. A special tactic that will help to keep the emphasis balanced is to make the assignments quite varied. As soon as is

feasible, have most of the big ideas of the chapter included in each assignment. Because this point has applications in other parts of the course, we illustrate it here by giving below a possible complete set of assignments for the chapter, including notes to show what topics are introduced on each day. The set of assignments is planned for 15 days. The left column, labelled, "Class Discussion," indicates the topic to be developed on that day. The right-hand column is the assignment for the given day; that is, in the row labelled "Day No. 2," the indicated assignment is to be made on Day No. 2, and will be due on Day No. 3.

Day No.	Class Discussion	Assignment
1	The general idea of a locus. Equation of circle with center (h, k) and radius r.	Read pp. 367-390 pp. 376-377 - 1, 3, 4, 5, 9, 14, 15, 21 p. 369 - 1, 2, 5, 10, 11
2	Continuation of locus idea, and especially of the intersection of loci and the algebraic technique involved.	p. 373 (A-1) - 1, 4, 5, 6, 9 pp. 376-377,- 16, 20, 22
3	Line tangent to a curve. Families of lines and curves. Parameters.	pp. 376-377 - 6, 19 p. 381 - 1, 2, 3, 6, 8, 9
4	How to solve various problems on page 381 (B).	Read pp. 382-384. p. 381 - 4, 10, 15 p. 385 (A) - 1, 2, 6, 7
5	Brief review of the ellipse, including Ex. 1 on p. 385 (B). The hyperbola (p. 386).	Read pp. 382-386 p. 382 - 16 p. 385 (B) - 2, 3 p. 387 (A) - 1, 2, 5 p. 387 (B) - 1
6	Asymptotes (pp. 388, 389) Summary (p. 390)	pp. 377-378 - 13, 20 pp. 381-382 - 5, 13, 17 p. 387 (B) - 2 p. 391 (A-1) - sketch 1-8; construct No. 9
7	The solution of two second-degree equations.	p. 381 - 7, 12 p. 391 (A-2) - 10, 11 p. 392 - 9 p. 393 (A-1) - 1, 3, 4, 9
8	Homogeneous equations of the second-degree	p. 382 - 18 p. 392 - 8 p. 393 - 2, 5 p. 396 (B-1) - 1 to 6

Day No.	Class Discussion	Assignment
9	Work Period or Test	p. 382 - 19 p. 396 (B-2) - 1, 3, 8 Read pp. 397-399
10	The conic sections (pp. 397-399)	p. 382 - 20 p. 400 - 1, 3, 4, 5, 6, 8, 12 p. 402 - 1, 12
11	Work Period	pp. 400-401 - 7, 11 14, 15 p. 402 - 2, 10, 13, 15
12	Work Period	p. 401 - 16 p. 402 - 3, 9, 16, 20 p. 403 - 7, 8, 11, 15
13	Work Period	p. 401 - 19 p. 402 - 4, 8, 22 p. 403 - 10, 16, 17, 18
14	Work Period	p. 404 - Chapter Test
15	Test (See Test Booklet)	

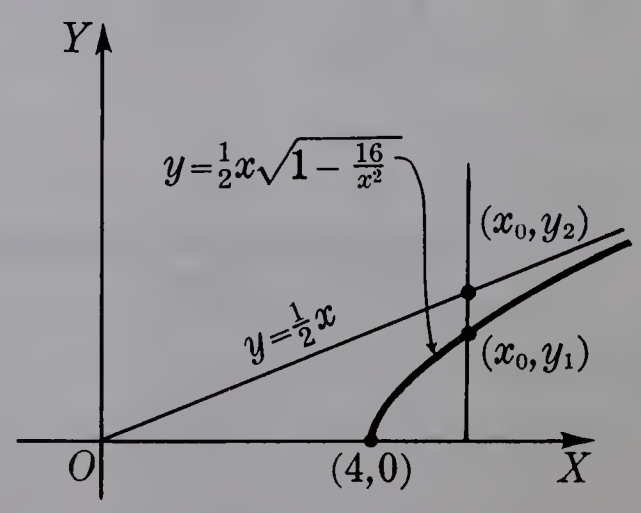
Notes on the above outline:

With no new topics scheduled for days 11 to 14, there is enough flexibility to make modifications suggested by questions from the class. Also, in these and other class periods, it may be possible to do some of the problems that are not listed in the assignments. It is to be hoped that the teacher will not always need the entire class period to develop some of the indicated topics, so that portions of most of the class periods will be available for work by the pupils.

Page 388—Asymptotes

In the teacher's discussion with the class of the table at the bottom of this page, a useful variation of the language of the text might be as follows:

As the line $x = x_0$ moves to the right, that is, as x_0 becomes as large as we please, we see that $y_2 - y_1$ becomes smaller and smaller. We may say that we can make $y_2 - y_1$ as small as we please by taking x_0 large enough. Such language connects



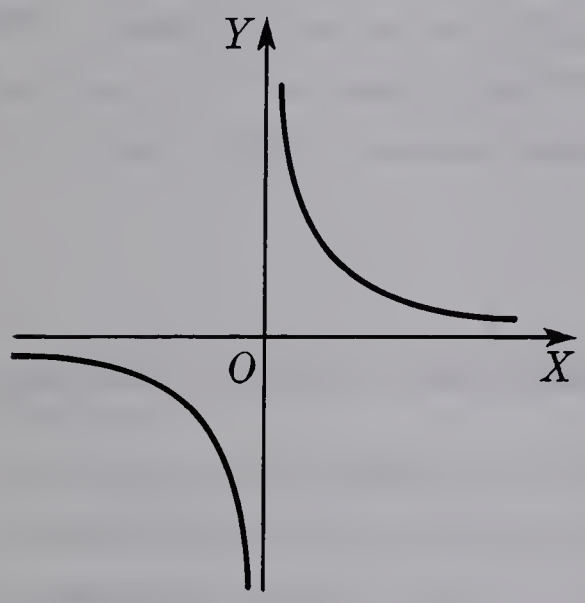
with the language on page 509. We have left it out of the text at this point because it seemed to put more time and attention on the concept than we thought was good pedagogy here. In class, the teacher might even want to introduce the usual notation as follows. If we want $|y_2 - y_1|$ to be less than a small positive number, ϵ, then we can choose a value, X, for x_0 such that whenever $x_0 > X$ then $|y_2 - y_1| < \epsilon$.

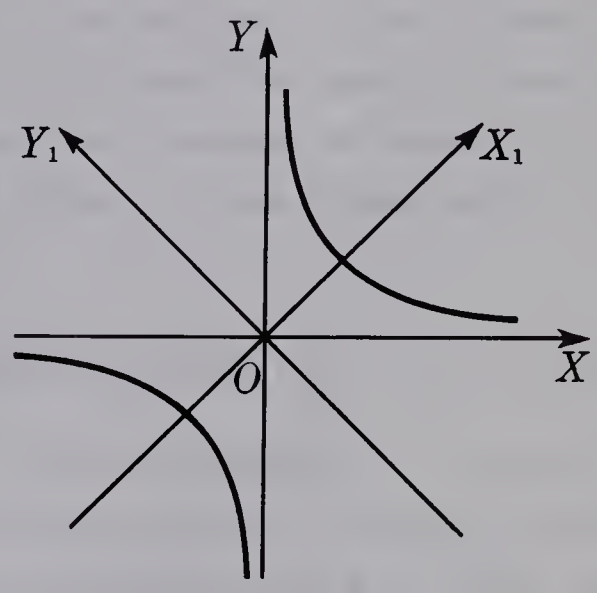
Page 390—The Equation xy = k

The teacher is reminded that xy = k may be put into the form $x^2 - y^2 = c$ by a rotation of the axes through 45°. The equations of rotation are: $x = x_1 \cos\theta - y_1 \sin\theta$

$y = x_1 \sin\theta + y_1 \cos\theta$

(See page 99 of this manual for the derivation of these equations.)





We carry out the rotation for xy = 2.

$(x_1 \cos 45° - y_1 \sin 45°)\ (x_1 \sin 45° + y_1 \cos 45°) = 2$

$\frac{1}{2}x_1^2 - \frac{1}{2}y_1^2 = 2$ or $x_1^2 - y_1^2 = 4$. Most classes will not know enough trigonometry at this point to pursue this matter further.

Page 392—Case 1

If one of the equations is of the form xy = k there is a method of procedure other than the one here presented. We illustrate with the same pair of equations:

$$x^2 + y^2 = 17 \qquad (A)$$
$$xy = 4 \qquad (B)$$

Multiply (B) by 2 and add the result to (A):

$$x^2 + 2xy + y^2 = 25$$

$x + y = 5$	$x + y = 5$
$xy = 4$	$xy = 4$
$x(5 - x) = 4$	$x(-5 - x) = 4$
$x^2 - 5x + 4 = 0$	$x^2 + 5x + 4 = 0$

$x =$	4	1		$x =$	-4	-1	
$y =$	1	4		$y =$	-1	-4	

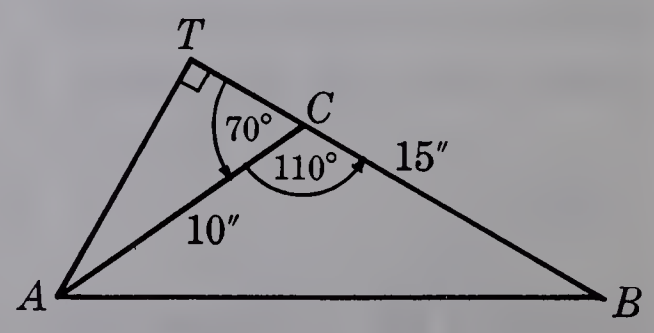
Pages 394-401

It is quite possible and reasonable to omit all of these pages if the class is just an ordinary class. Some of the problems here are too specialized and some are too hard. They are all interesting if the class can deal with them without undue strain. The technical skill required for some of the exercises on page 402 is considerable and must be practiced if it is to be learned. The exercises on page 400 are not too difficult for an ordinary class. They are included among the B exercises because they lead up to some of the exercises on page 401.

Chapter 15. The Sine and Cosine Functions. Oblique Triangles

Chapter 11 on measurement in right triangles and Chapter 15 on oblique triangles provide, in conjunction with the information picked up in a plane geometry course, all the tools necessary for answering numerical questions about a triangle when sufficient data are given to determine the triangle. Actually, as soon as the trigonometric functions are defined for acute angles, it is possible to answer the questions. The wider base developed in Chapter 15 merely makes the methods of answering the questions more direct—not necessarily easier. The development of these more direct methods is the basic motivation for Chapter 15. To illustrate this point we consider the following problem:

Given triangle ABC with $\angle C = 110°$, $b = 10$ in., $a = 15$ in. Find c, the length of side AB.



First solution (Using Chapter 11 only): Construct the altitude AT.

Then, $\angle TCA = 70°$

$TA = 10 \sin 70° = 9.397$

$TC = 10 \cos 70° = 3.42$

$$TB = 3.42 + 15 = 18.42$$

$$AB = c = \sqrt{(9.397)^2 + (18.42)^2}$$

$$c = \underline{20.7}$$

Second solution (Using Chapter 15)

By the Law of Cosines we have

$$c^2 = 10^2 + 15^2 - 2 \cdot 10 \cdot 15 \cdot \cos 110°$$

$$c = \sqrt{325 + 300 \cdot \cos 70°} = \underline{20.7}$$

Students at this stage in their study of mathematics are highly intrigued by the challenge set up when the proposition is asserted that one should be able to answer all questions of a numerical nature about a triangle whenever sufficient data are given to determine the triangle. The congruence theorems of geometry tell us that a triangle is determined if its three sides are known, or two sides and the included angle, or two angles and the included side. When the three sides are given, we find one angle by using the Law of Cosines and then find a second angle by using the Law of Sines. The following examples show how to answer a few questions about $\triangle ABC$ if a = 5 in., b = 8 in., c = 7 in.

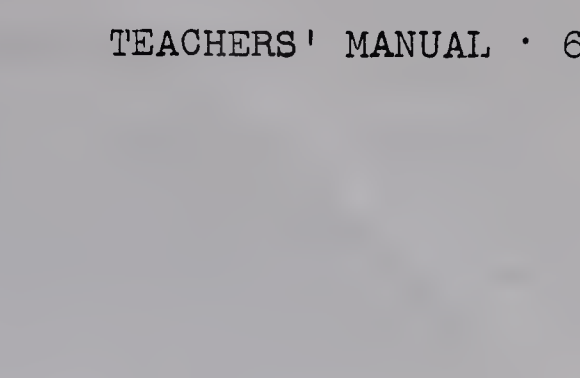
(1) Angles

By the Law of Cosines,

$$8^2 = 5^2 + 7^2 - 2 \cdot 5 \cdot 7 \cdot \cos B$$

$$\cos B = \frac{1}{7} = 0.1429$$

$$\underline{\angle B = 81°47'}.$$



Since $\angle B$ is opposite the longest side, then $\angle B$ must be the largest angle of the triangle. Since cos B > 0, then $\angle B$ must be acute. The angles A and C are also acute since they are smaller than $\angle B$, and so, from the Law of Sines, $\frac{\sin A}{5} = \frac{\sin B}{8}$.

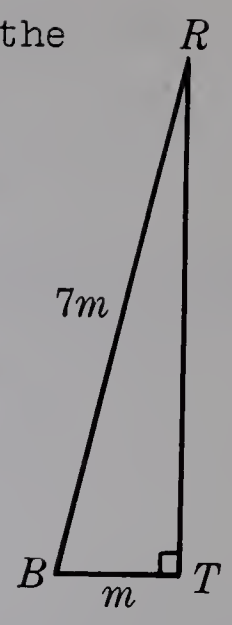
$$\sin A = \frac{5}{8} \sin 81°47' = 0.6186$$

$$\underline{\angle A = 38°13'}$$

Then $$C = 180 - (\angle B + \angle A) = 60°$$

Note. If we first obtain $\angle A$ by the Law of Cosines and then try to get $\angle B$ from the Law of Sines, we end up with sin B = 0.9898, and must still decide, by recourse to the Law of Cosines, whether $\angle B$ is acute or obtuse.

Reference triangle. Since we have determined in the $\triangle ABC$ that $\cos B = \frac{1}{7}$, then we may use a reference triangle to determine the values of the other functions of $\angle B$.



$RT = \sqrt{49m^2 - m^2} = 4m\sqrt{3}$

So $\underline{\sin B = \frac{4m\sqrt{3}}{7m} = \frac{4}{7}\sqrt{3}}$, etc.

(2) Area (K) (see figure, p. 69)

Since $K = \frac{1}{2}ac \sin B$

then $\underline{K = \frac{1}{2} \cdot 5 \cdot 7 \cdot \frac{4}{7}\sqrt{3} = 10\sqrt{3}}$

(3) Altitudes (h_a, h_b, h_c) (see figure, p. 69)

Since the area is known, $(10\sqrt{3})$, then $\frac{1}{2} \cdot 8 \cdot h_b = 10\sqrt{3}$

$h_b = \frac{5}{2}\sqrt{3}$

Similarly, $\underline{h_a = 4\sqrt{3}}$ and $\underline{h_c = \frac{20}{7}\sqrt{3}}$

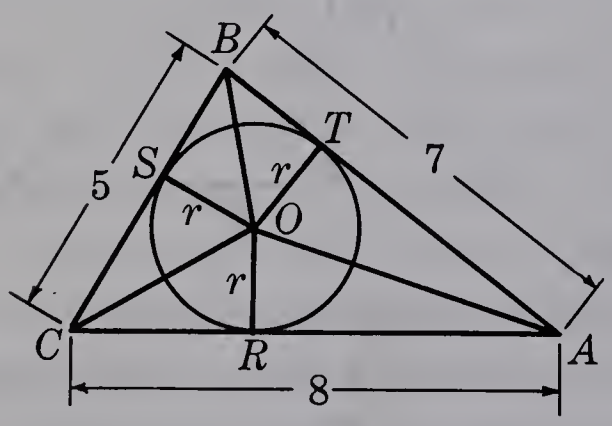
(4) Radius of the inscribed circle (r)

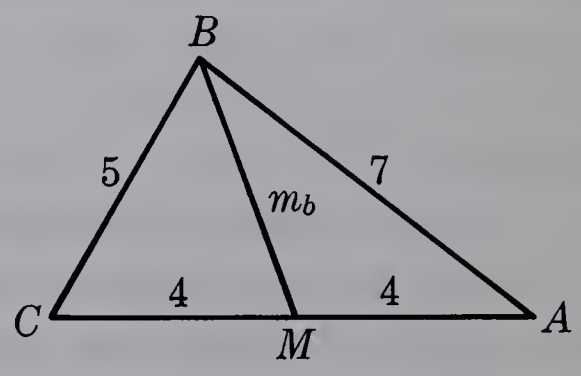
The area of $\triangle ABC$ may be expressed in terms of r:

$\frac{1}{2} \cdot 8 \cdot r + \frac{1}{2} \cdot 7 \cdot r + \frac{1}{2} \cdot 5 \cdot r = 10r$

But the area of $\triangle ABC$ is also equal to $10\sqrt{3}$. $10r = 10\sqrt{3}$

$\underline{r = \sqrt{3}}$





(5) Median to side b. (m_b)

In $\triangle ACB$, $\cos C = \frac{5^2 + 8^2 - 7^2}{2 \cdot 5 \cdot 8} = \frac{1}{2}$

In $\triangle MCB$, $\cos C = \frac{5^2 + 4^2 - (m_b)^2}{2 \cdot 5 \cdot 4} = \frac{41 - (m_b)^2}{40}$

$\frac{41 - (m_b)^2}{40} = \frac{1}{2}$

$\underline{m_b = \sqrt{21}}$

(6) Radius of the circumscribed circle

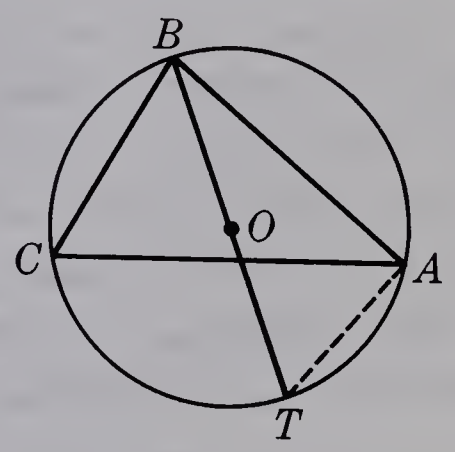
Even a slow class can be intrigued by the fact that the diameter of the circumscribed circle is the constant of proportionality in the Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

In the triangle under consideration (see figure, p. 69), since $\sin B = \frac{4}{7}\sqrt{3}$, we have $8 \div \frac{4}{7}\sqrt{3} = 2R$ whence $\underline{R = \frac{7}{3}\sqrt{3}}$.

We included the demonstration of this relation as a B exercise (p. 424, Ex. 8) because a class that has not had some geometry of the circle would have difficulty with it.

Given the circle O with $\triangle ABC$ inscribed in it. Since we wish to reach a conclusion that involves the diameter, we draw the diameter BT, then draw TA. In $\triangle BAT$, $\sin T = \frac{c}{2R}$. But $\angle T = \angle C$, since they each have the same measure as half the number of degrees in $\overset{\frown}{BA}$.



Therefore, $\sin C = \frac{c}{2R}$

Whence, $\frac{c}{\sin C} = 2R.$

One can tie this relationship into a general review of ratio and proportion. (See pages 124-129 and 193-204 of the text.) If the class has studied geometry this is also a good time to point out again that the phrases "constant of proportionality" and "ratio of similitude" are arithmetically equivalent. We merely use one rather than the other depending on the context of the discussion.

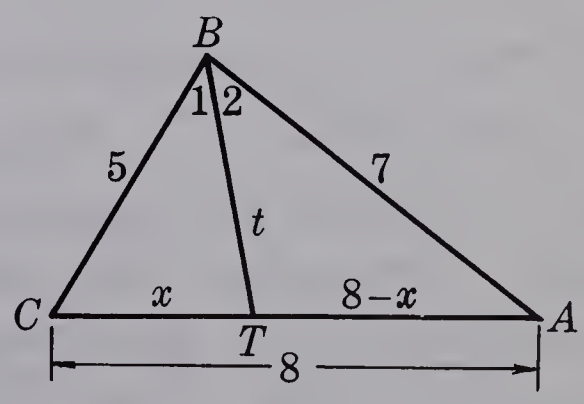
(7) Bisector of largest angle

Since BT bisects $\angle CBA$,

we have $\frac{x}{8 - x} = \frac{5}{7}$

Whence $x = \frac{10}{3}$

$8 - x = \frac{14}{3}$



(See Theorem 41, page 217, of A Course in Geometry

By the Law of Cosines we now may write

$$\begin{cases} \left(\frac{10}{3}\right)^2 = 5^2 + t^2 - 2 \cdot 5 \cdot t \cdot \cos B_1 & (1) \\ \left(\frac{14}{3}\right)^2 = 7^2 + t^2 - 2 \cdot 7 \cdot t \cdot \cos B_2 & (2) \end{cases}$$

Since $B_1 = B_2$ we may make the last terms on the right equal by multiplying ① by 7 and ② by 5:

$$\begin{cases} 7\left(\frac{10}{3}\right)^2 = 7 \cdot 5^2 + 7t^2 - 2 \cdot 5 \cdot 7 \cdot t \cos B_1 & ③ \\ 5\left(\frac{14}{3}\right)^2 = 5 \cdot 7^2 + 5t^2 - 2 \cdot 5 \cdot 7 \cdot t \cos B_2 & ④ \end{cases}$$

Subtracting ④ from ③,

$$7\left(\frac{10}{3}\right)^2 - 5\left(\frac{14}{3}\right)^2 = 7 \cdot 5^2 - 5 \cdot 7^2 + 2t^2$$

Whence, $t = \frac{5}{3}\sqrt{7}$.

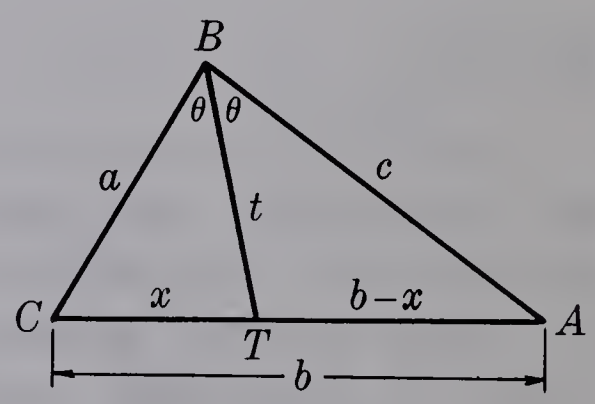
The general solution of $\triangle ABC$ in terms of a, b, c.

A great many interesting variations in the approach to the solution of the triangle are available. We regard them as suitable for special projects, designed primarily for the good student. Their investigation as a special project may be justified on the basis of entertainment, if on no other basis.

For example, the computation of the length of the angle-bisector may be approached in general terms as follows.

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\frac{x}{b - x} = \frac{a}{c}, \text{ and } x = \frac{ab}{a + c}.$$



$$\cos C = \frac{a^2 + x^2 - t^2}{2ax}$$

$$\therefore \frac{a^2 + b^2 - c^2}{2ab} = \frac{a^2 + \frac{a^2b^2}{(a + c)^2} - t^2}{\frac{2a^2b}{(a + c)}}$$

$$\text{and } t = \frac{1}{a + c}\sqrt{2a^2c^2 + a^3c - ab^2c + ac^3}$$

In terms of the semiperimeter $[s = \frac{1}{2}(a + b + c)]$, $t = \frac{2}{a + c}\sqrt{acs(s - b)}$. The technical manipulations involved here are not trivial and should not be lightly undertaken by an eleventh grader. Of course, this derivation can be based merely on a couple of applications of the Pythagorean theorem and on the angle-bisector theorem. This is obvious, since the Law of Cosines itself is based on the Pythagorean theorem.

Derivation of $R = \frac{abc}{4K}$ (See exercises 8 and 9 on page 424.)

This extraordinary relation says that the radius of the circle that circumscribes $\triangle ABC$ is equal to the product of the three

sides divided by 4 times the area of the triangle! It is hard to imagine a more useless formula, but students are intrigued by it. Use it as "frosting on the cake" or as a light touch to brighten up a dull Friday afternoon. To derive it one may proceed as follows.

$$\text{Since } 2R = \frac{a}{\sin A}$$

$$\text{and} \quad K = \tfrac{1}{2}bc \sin A,$$

$$\text{Then} \quad 2R = \frac{a}{\frac{2K}{bc}}$$

$$\text{or} \quad R = \frac{abc}{4K}$$

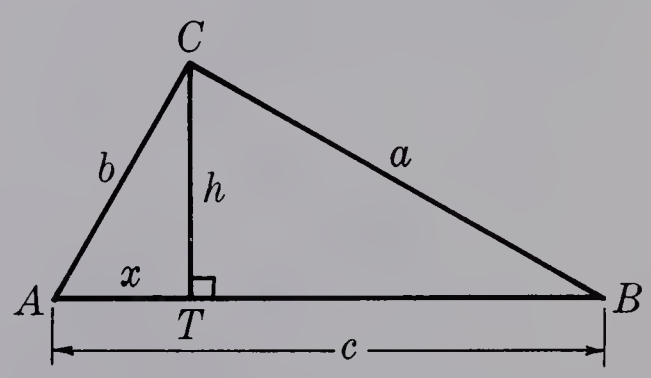
Since K is known in terms of the sides, then R is now known in terms of the sides.

The study of the triangle may be undertaken in analytic form. In employing this approach, there is a need for considerable trigonometric information of the kind developed in Chapter 19. On the other hand, the student may derive, without introducing a coordinate system, general formulas for various quantities with no more trigonometric information than the Law of Sines and the Law of Cosines. We illustrate this point below.

(1) Area

$$K = \tfrac{1}{2}ch$$

$$x = b \cos A$$



$$\text{But } \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\therefore x = \frac{b^2 + c^2 - a^2}{2c}$$

$$h = \sqrt{b^2 - \frac{(-a^2 + b^2 + c^2)^2}{4c^2}}$$

$$= \frac{1}{2c}\sqrt{4b^2c^2 - (-a^2 + b^2 + c^2)^2}$$

$$= \frac{1}{2c}\sqrt{(2bc + b^2 + c^2 - a^2)(2bc + a^2 - b^2 - c^2)}$$

$$= \frac{1}{2c}\sqrt{(b + c + a)(b + c - a)[a^2 - (b^2 - 2bc + c^2)]}$$

$$= \frac{1}{2c}\sqrt{(b + c + a)(b + c - a)(a + b - c)(a - b + c)}$$

If now, we put $a + b + c = 2s$

then $-a + b + c = 2(s - a)$

$a - b + c = 2(s - b)$

$a + b - c = 2(s - c)$

and so

$$h = \frac{1}{2c}\sqrt{2s \cdot 2(s - a) \cdot 2(s - b) \cdot 2(s - c)}$$

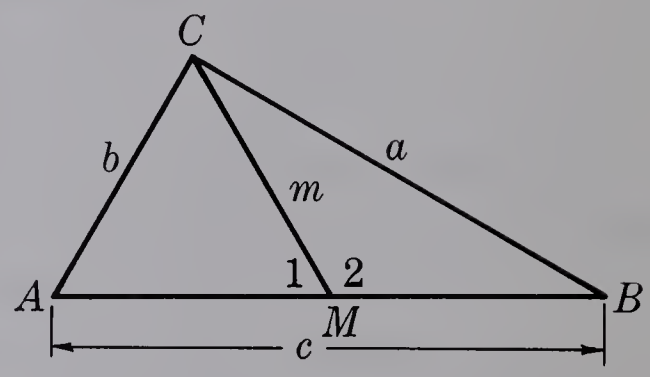
$$h = \frac{2}{c}\sqrt{s(s - a)(s - b)(s - c)}$$

$$\therefore K = \frac{1}{2}ch = \frac{c}{2} \cdot \frac{2}{c}\sqrt{s(s - a)(s - b)(s - c)}$$

$$K = \sqrt{s(s - a)(s - b)(s - c)}$$

(2) Median to side C

$$\begin{cases} b^2 = m^2 + \frac{1}{4}c^2 - 2 \cdot m \cdot \frac{1}{2}c \cos M_1 \\ a^2 = m^2 + \frac{1}{4}c^2 - 2 \cdot m \cdot \frac{1}{2}c \cos M_2 \end{cases}$$

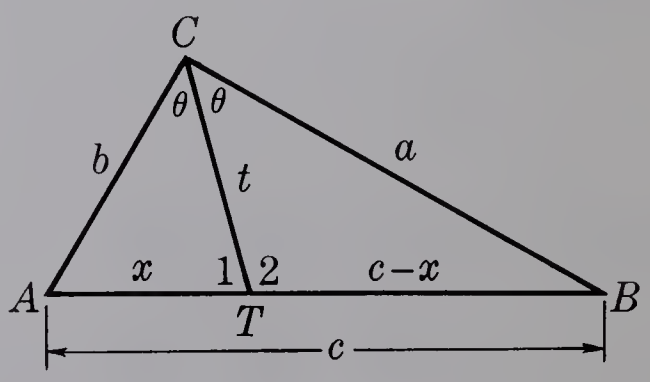


$\cos M_1 = -\cos M_2$

$$\therefore b^2 + a^2 = 2m^2 + \frac{1}{2}c^2$$

$$m = \sqrt{\frac{1}{2}\left(a^2 + b^2 - \frac{1}{2}c^2\right)}$$

(3) Bisector of $\angle C$



$$\frac{x}{c - x} = \frac{b}{a}$$

$$ax = bc - bx$$

$$x = \frac{bc}{a + b}$$

$$c - x = c - \frac{bc}{a + b} = \frac{ac}{a + b}$$

$$\begin{cases} b^2 = t^2 + \left(\frac{bc}{a + b}\right)^2 - 2t\left(\frac{bc}{a + b}\right) \cos T_1 \\ a^2 = t^2 + \left(\frac{ac}{a + b}\right)^2 - 2t\left(\frac{ac}{a + b}\right) \cos T_2 \end{cases}$$

$T_1 = 180 - T_2$. $\therefore \cos T_1 = -\cos T_2$.

$$\frac{b^2 - t^2 - \left(\frac{bc}{a + b}\right)^2}{a^2 - t^2 - \left(\frac{ac}{a + b}\right)^2} = -\left(\frac{bc}{a + b}\right) \cdot \left(\frac{a + b}{ac}\right)$$

$$ab^2 - at^2 - a\left(\frac{bc}{a + b}\right)^2 = -ba^2 + bt^2 + b\left(\frac{ac}{a + b}\right)^2$$

$$(b + a)t^2 = ab(b + a) - \frac{abc^2}{(a + b)^2}[b + a]$$

$$t = \sqrt{ab - \frac{abc^2}{(a+b)^2}}$$

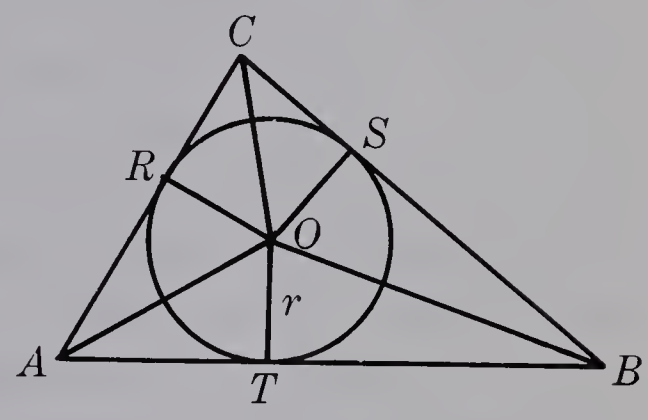
$$t = \frac{1}{a+b}\sqrt{ab(a^2 + 2ab + b^2 - c^2)}$$

$$= \frac{1}{a+b}\sqrt{ab \cdot 2s \cdot 2(s-c)} = \frac{2}{a+b}\sqrt{abs(s-c)}$$

(4) Radius of inscribed circle

If "$\triangle ABC$" refers to the "area of $\triangle ABC$," etc., then

$\triangle ABC = \triangle AOB + \triangle AOC + \triangle BOC$



$$K = \sqrt{s(s-a)(s-b)(s-c)}$$

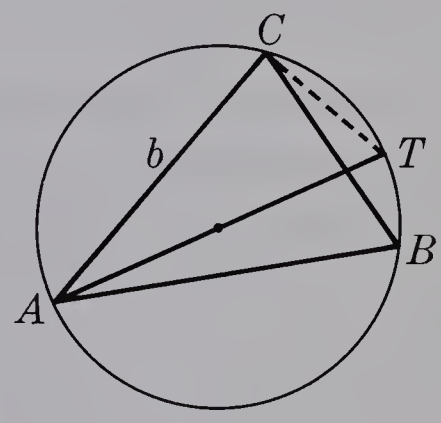
$$= \frac{1}{2}rc + \frac{1}{2}rb + \frac{1}{2}ra$$

$$= r\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c\right)$$

$$= rs$$

$$\therefore r = \frac{K}{s} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$$

(5) Radius of circumscribed circle



Draw diameter AT

$\angle CTA = \angle CBA$

$$\sin \angle CBA = \sin \angle CTA = \frac{b}{2R}$$

$$\therefore 2R = \frac{b}{\sin B}$$

$$\text{or } R = \frac{b}{2} \cdot \frac{1}{\sin B}$$

$$\text{But } K = \frac{1}{2}ac(\sin B) = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\text{Whence, } \sin B = \frac{2}{ac}\sqrt{s(s-a)(s-b)(s-c)}$$

$$\therefore R = \frac{b}{2} \cdot \frac{ac}{2\sqrt{s(s-a)(s-b)(s-c)}}$$

$$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

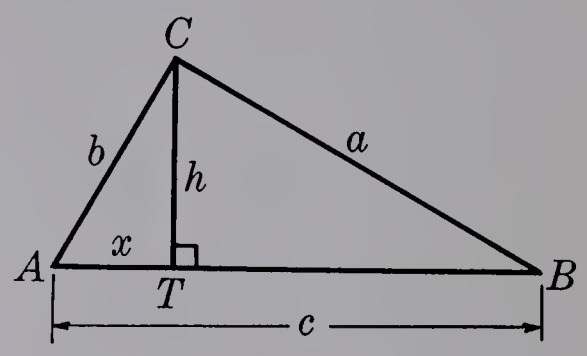
Chapter 16. Trigonometric Equations and Identities

Chapters 15 and 16 are really parts of the same theme: the definition of the trigonometric functions of angles of any size. Chapter 16 may be regarded as a continuation of the development of Chapter 15. If the teacher wishes to introduce only enough of the trigonometric relations to make convenient the complete solution of any triangle, then only Chapter 15 need be done.

If all the trigonometry is to be done in one year we suggest that frequent mention be made in Chapter 16 of the unit circle. Point out frequently that no loss of generality occurs if r is taken as 1. If there seems any point in doing so, the teacher may bring in, during the study of Chapter 16, the definitions of the functions as developed on pages 532-535.

A fine motivation for the study of the relations of Chapter 16 can be achieved by an approach something like the following.

The Law of Cosines tells us that $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$. It is plain that $h = b \sin A$. So $\underline{h}$ could be found in terms of a, b, c if there were a connection between sin A and cos A. This relationship is $\sin^2 A + \cos^2 A = 1$ for any A whatever. The relationship may be derived as indicated below and then used in stating the formula for $\underline{h}$ in terms of a, b, c.



Derivation of $\sin^2 \theta + \cos^2 \theta = 1$, for every θ.

If P (x, y) represents any point in the Cartesian plane and $r = |PO|$, then it is clear from the Pythagorean theorem that $x^2 + y^2 = r^2$. It follows that

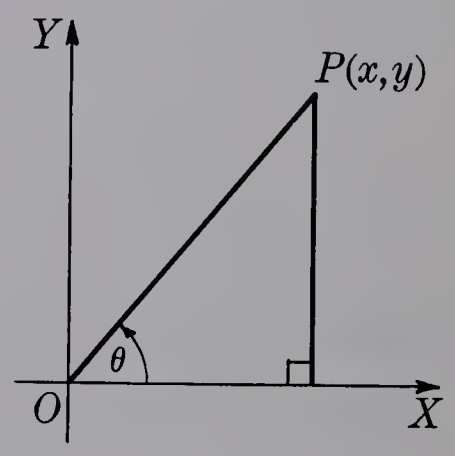
$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1.$$

But $$\frac{x}{r} = \cos \theta$$

$$\frac{y}{r} = \sin \theta$$

and so we have $\cos^2 \theta + \sin^2 \theta = 1$,

or $\sin^2 \theta + \cos^2 \theta = 1$.



The Formula for h in terms of a, b, c.

Hence, if $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

then $$\sin A = \sqrt{1 - \frac{(b^2 + c^2 - a^2)^2}{(2bc)^2}}$$

$$= \frac{1}{2bc}\sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2}$$

$$= \frac{1}{2bc}\sqrt{(b + c + a)(b + c - a)(a + b - c)(a - b + c)}$$

If $2s = a + b + c$,

Then $\sin A = \frac{1}{2bc}\sqrt{2s(2s - 2a)(2s - 2b)(2s - 2c)}$

$$\sin A = \frac{2}{bc}\sqrt{s(s - a)(s - b)(s - c)}$$

Now $h = b \sin A$

$$\therefore h = \frac{2}{c}\sqrt{s(s - a)(s - b)(s - c)}$$

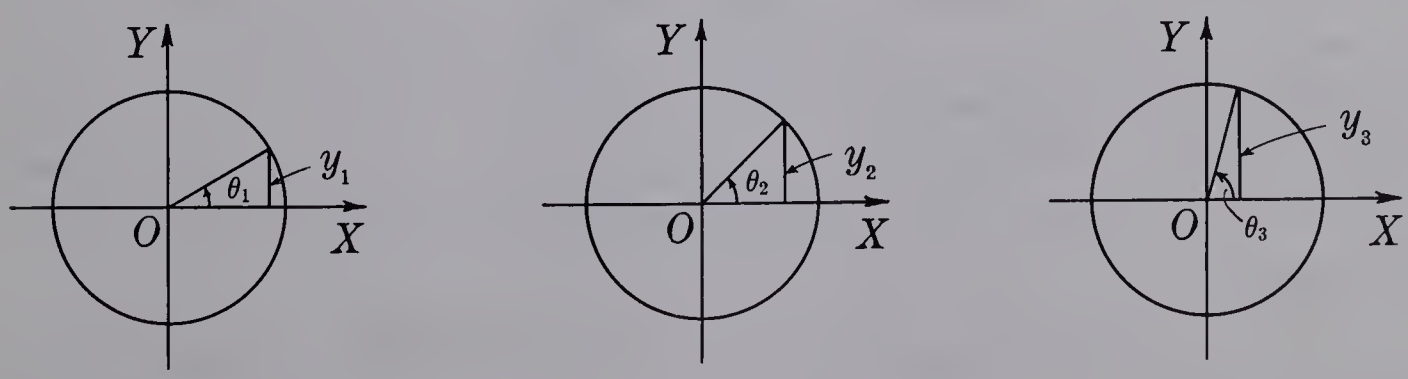
There are many obscure relationships among the various parts of a triangle. Some of them require a knowledge of the trigonometric identities presented in Chapter 19 as well as of those presented in Chapter 16. The interested teacher will find a definitive discussion in Chapters X, XI, and XII of A Treatise on Plane and Advanced Trigonometry by E. W. Hobson. The recondite nature of many of the relations discussed by Hobson makes them wholly unsuitable for general education in mathematics at this stage, but they are certainly interesting and may fascinate some bright student. To illustrate the reconditeness we merely quote one of the simpler problems:

"If t_1, t_2, t_3 represent the lengths of the tangents from the centers of the escribed circles to the circumcircle of $\triangle ABC$, prove that $\frac{1}{t_1^2} + \frac{1}{t_2^2} + \frac{1}{t_3^2} = \frac{a + b + c}{abc}$."

(An escribed circle of a triangle is tangent to one of the sides of the triangle and to the extensions of the other two sides).

The various discussions of the trigonometric functions, in Chapters 11, 15, 16, 19, sometimes become a little confused in the pupils' mind. An effective way to clarify things for him is to summarize them in one presentation—after the presentations of the book have been taken up in order. The summary might proceed somewhat as follows:

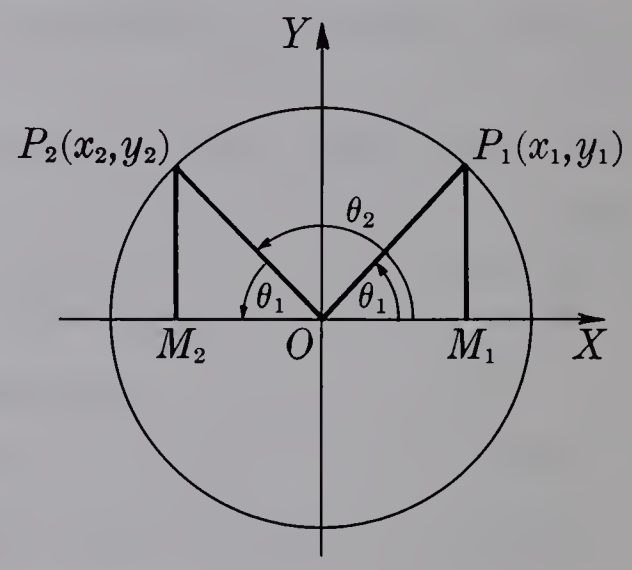
If we draw various positions of θ on a given unit circle



(radius = 1), then, to every value of θ less than 90°, there will

correspond a value of y, and hence, a value of the ratio $\frac{y}{r}$. If θ changes, so does y, but the value of r does not change. Hence, y depends on θ, and the set of ordered pairs (θ, y) is a function. Since we are dealing with a unit circle, $\sin \theta = \frac{y}{1} = y$, and $y = \sin \theta$ defines the function. Similarly, $x = \cos \theta$, and $\frac{y}{x} = \tan \theta$. For $0° \leq \theta \leq 90°$, the set of values for y, x, and $\frac{y}{x}$ have been computed for us, and are recorded in tables headed, respectively, natural sines, natural cosines and natural tangents.

When θ is not acute, then we can always find an acute angle which has the same sine, cosine, or tangent as θ except for a suitable choice of sign. (The x is positive in Quadrants 1 and 4, negative in Quadrants 2 and 3, and the y is positive in Quadrants 1 and 2, negative in Quadrants 3 and 4.) To find this acute angle we proceed as follows.



Since, by definition,

$y_2 = \sin \theta_2$,

$x_2 = \cos \theta_2$,

we require an acute angle, θ_1, such that

$\sin \theta_1 = |y_2|$, $\cos \theta_1 = |x_2|$.

This is plainly $\angle M_2OP_2$. If we place this angle in the position M_1OP_1, then $\triangle OP_1M_1 \cong \triangle OP_2M_2$ since $OP_1 = OP_2$, $\angle P_1OM_1 = \angle P_2OM_2$, $\angle OM_1P_1 = \angle OM_2P_2$.

Then $|y_1| = |y_2|$ and $|x_1| = |x_2|$. Hence

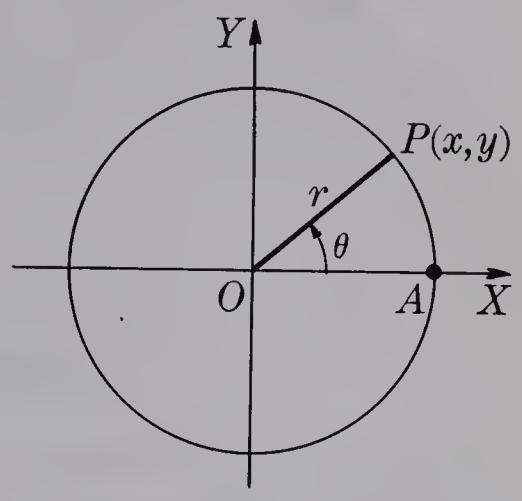
$$\sin \theta_2 = y_2 = y_1 = \sin \theta_1$$
$$\cos \theta_2 = x_1 = -x_2 = -\cos \theta_1$$

In a similar manner we can associate with every θ_i an acute angle, θ_1, such that the coordinates of P_i will always be equal to the absolute values of the coordinates of P_i.

Finally, (Chapter 19), we see that to every position of P there corresponds a length of arc, s, on the unit circle so that we may say $y = \sin s$, $x = \cos s$, and these correspondences are not directly dependent on the notion of angle or of the measure of an angle. If Chapter 19 is not done then no useful purpose is served by divorcing the independent variable from the notion of angle measure.

Page 439. Exercises 13-16

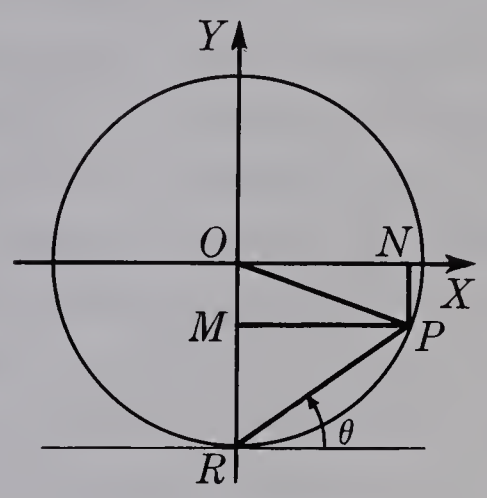
Students always wish to know how the parameter θ in these problems is selected. The answer, of course, is that it may be selected in a variety of ways. For example, consider the circle with radius r and center at the origin. If θ is chosen as indicated in the upper figure, then it is clear that $x = r \cos \theta$ and $y = r \sin \theta$. If θ is chosen as indicated in the lower figure, then $\angle POR = 2\theta$ and, from $\triangle POM$,



$PM = x = r \sin 2\theta.$

$|OM| = |PN| = |y| = |r \cos 2\theta|$

or $y = -r \cos 2\theta.$



Hence, $\begin{cases} x = r \sin 2 \\ y = -r \cos 2 \end{cases}$

are parametric equations of the circle with center at the origin and radius of r, with θ located as indicated in the lower figure.

Expressing each trigonometric function in terms of any one of the functions.

A good exercise to review the relations of this chapter is to express all of the six trigonometric functions in terms of a single one of the functions. To illustrate, we derive formulas for the functions in terms of the tangent.

(1) $\tan \theta = \tan \theta$

(2) $\cot \theta = \dfrac{1}{\tan \theta}$

(3) Since $1 + \tan^2 \theta = \sec^2 \theta$, then $\sec \theta = \pm\sqrt{1 + \tan^2 \theta}$

(4) $\cos \theta = \dfrac{1}{\sec \theta} = \dfrac{1}{\pm\sqrt{1 + \tan^2 \theta}}$

(5) $\sin \theta = \pm\sqrt{1 - \cos^2 \theta} = \pm\sqrt{1 - \dfrac{1}{1 + \tan^2 \theta}} = \pm \dfrac{\tan \theta}{\sqrt{1 + \tan^2 \theta}}$

(6) $\csc \theta = \dfrac{1}{\sin \theta} = \pm \dfrac{\sqrt{1 + \tan^2 \theta}}{\tan \theta}$

The six functions may be expressed in terms of the sine:

(1) $\sin \theta = \sin \theta$ (2) $\cos \theta = \pm\sqrt{1 - \sin^2 \theta}$

$$(3)\ \csc\theta = \frac{1}{\sin\theta} \qquad (4)\ \sec\theta = \frac{1}{\pm\sqrt{1-\sin^2\theta}}$$

$$(5)\ \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\pm\sqrt{1-\sin^2\theta}}$$

$$(6)\ \cot\theta = \frac{\pm\sqrt{1-\sin^2\theta}}{\sin\theta}$$

Chapter 17. Polynomials

In one's introductory remarks about this chapter it is helpful to point out the ways in which it may be regarded as an extension of Chapter 13 on the quadratic function. That is, in the polynomial of degree n: there are n roots, the coefficients and the roots are related, the polynomial has values which may be regarded as extreme values, and, in general, the polynomial may be factored into n linear factors. It should be pointed out however, that the difficulty of determining roots, factors, extreme values, etc. increases greatly as the degree of the polynomial increases.

Our first problem in the study of the cubic, as with our study of the quadratic, is the problem of changing from one form of an expression to another. Usually this involves changing the polynomial form to the factored form. The solution for the quadratic is of course well known by this time and is simple. But the factor theorem is needed for the cubic and this must be preceded by the division identity and the remainder theorem.

An interesting sidelight on this problem of form is the rearrangement of the polynomial that is used when the evaluation of it is to be carried out on a computing machine. In programming the polynomial, $ax^4 + bx^3 + cx^2 + dx + e$, we rearrange it as follows:

$$\begin{aligned} & ax^4 + bx^3 + cx^2 + dx + e \\ &= x(ax^3 + bx^2 + cx + d) + e \\ &= x[x(ax^2 + bx + c) + d] + e \\ &= x(x[x(ax + b) + c] + d) + e \end{aligned}$$

This arrangement may be verbalized by saying:

> The first coefficient is multiplied by x; then the second coefficient is added to the product;
> this sum is multiplied by x and then the third coefficient is added;

this sum is multiplied by x and then the fourth coefficient is added; etc.

It is illuminating to carry out before the class an evaluation exercise using this arrangement. Thus if $f(x) = 2x^4 + 3x^3 - 5x^2 + 7x + 8$, find $f(3.5)$.

$$\begin{aligned} & 2x^4 + 3x^3 - 5x^2 + 7x + 8 \\ &= x(2x^3 + 3x^2 - 5x + 7) + 8 \\ &= x[x(2x^2 + 3x - 5) + 7] + 8 \\ &= x(x[x(2x + 3) - 5] + 7) + 8 \end{aligned}$$

If $x = 3.5$, we have

$$\begin{aligned} 7 + 3 &= 10 \\ 10(3.5) &= 35 \\ 35 - 5 &= 30 \\ 30(3.5) &= 105.0 \\ 105 + 7 &= 112 \\ 112(3.5) &= 392 \\ 392 + 8 &= \underline{400} \end{aligned}$$

Compare this with

$$2(3.5)^4 + 3(3.5)^3 - 5(3.5)^2 + 7(3.5) + 8$$

Etc.

or, later in the chapter, with the use of the remainder theorem and synthetic division:

2	+ 3	− 5	+ 7	+ 8	| 3.5
	7	35	105.0	392	
2	10	30	112	400	

The obvious similarity between the latter method and the first has led to calling the first method "the method of synthetic substitution."

Further points of similarity or contrast between the quadratic and the cubic are:

(a) The graph of a quadratic is symmetric about a line through its vertex; the graph of the cubic is symmetric about a point but no portion of it is ever symmetrical about a vertical line.

(b) The extreme value of the quadratic is easily determined by simple algebraic procedures; the extreme values of the cubic, if they exist, need the methods of the calculus for their easy determination.

(c) In both the quadratic and in the cubic, if the coefficients are real numbers and if $a + b_i$ is a root then $a - b_i$ is also a root. Similarly, if the coefficients are rational numbers, and if $p + \sqrt{q}$ is a root, then $p - \sqrt{q}$ is a root. Since these statements are obviously true of the quadratic, $ax^2 + bx + c = 0$, since its roots are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we placed no special emphasis on them in Chapter 13. However in the study of the cubic it is convenient to bring them explicitly into the open.

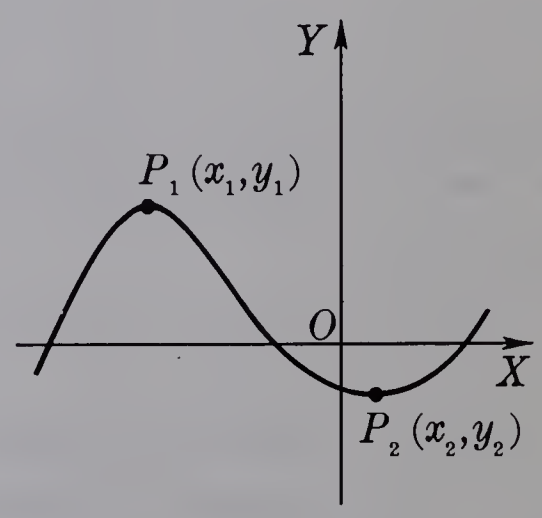
(d) The generalizations of the relationships between the roots and the coefficients are interesting and exciting even though they may be regarded as a little recondite.

(e) The basic purpose of the sequence of the chapter is to arrive, by way of a firm foundation, at a method for finding, to any desired degree of accuracy, an irrational root of a higher degree equation. There is an important distinction here between our elementary study of the quadratic and of the cubic functions. We can take the final step of finding the extreme value of the quadratic. For the cubic this step requires the differential calculus and so we find our study of the cubic necessarily terminated earlier than our study of the quadratic.

We recall for the teacher the problem of finding the relative maximum and the relative minimum of a cubic function by means of the calculus.

Given the function f defined by $y = f(x) = \frac{1}{3}x^3 + 2x^2 - 5x - 12$. If, for all x_i in the neighborhood of P_1, $y_1 > y_i$, then y_1 is a relative maximum of f.

Similarly, if, for all x_i in the neighborhood of P_2, $y_2 < y_i$, then y_2 is a relative minimum of f. At P_1 and P_2 the slope of the curve is 0 and therefore the first derivatives of f at these points are zero.



So, $y' = f'(x) = x^2 + 4x - 5 = 0$

whence $(x + 5)(x - 1) = 0$

$x = 1, -5$

If $x = x_2 = 1$, then
$f(x_2) = y_2 = \frac{1}{3} + 2 - 5 - 12 = -14\frac{2}{3}$, is a relative minimum.

If $x = x_1 = 5$, then
$f(x_1) = \frac{125}{3} + 50 - 25 - 12 = 54\frac{2}{3}$, is a relative maximum.

Page 451

Every cubic is symmetric about a point. If the point is not the origin then the axes can be translated so that the point of symmetry is the new origin. The coordinates of the point of symmetry can then be read with reference to the original axes. We illustrate with the curve defined by $y = x^3 - 6x + 3$.

We use the method of translating the axes so that the new equation has only odd powers of x. If we denote the variables referred to the new axes by x' and y', then

$$\begin{cases} x = x' + h \\ y = y' + k \end{cases} \quad \text{and} \quad \begin{cases} x' = x - h \\ y' = y - k \end{cases}$$

So, $y' + k = (x' + h)^3 - 6(x' + h) + 3$

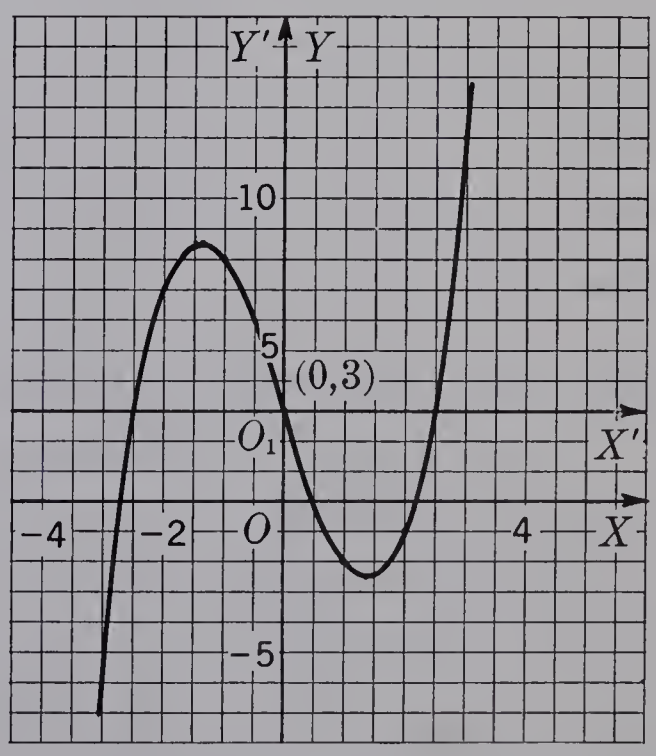
$$y' = (x')^3 + 3(x')^2h + 3x'h^2 + h^3 - 6x' - 6h + 3 - k$$
$$= (x')^3 + 3(x')^2h + (3h^2 - 6)x' + h^3 - 6h + 3 - k \quad (A)$$

Since this equation is to have only odd powers of x, we let the coefficients of the $(x')^2$ and $(x')^0$ terms equal zero:

$+3h = 0$ $\qquad$ $(h^3 - 6h + 3 - k) = 0$
$\therefore h = 0$ $\qquad$ $0 + 3 - k = 0$
$\therefore k = +3$

Substituting $h = 0$, $k = +3$ in (A) we have $y' = (x')^3 - 6x'$

The coordinates of the center of symmetry, referred to the original axes, are $(h, k) = (0, +3)$.



A second illustration follows:

$$y = x^3 - x^2 - 7x + 3$$

$x = x' + h$	$x' = x - h$
$y = y' + k$	$y' = y - k$

$y' + k = (x' + h)^3 - (x' + h)^2$
$\qquad - 7(x' + h) + 3$
$= (x')^3 + 3(x')^2h + 3x'h^2 + h^3 - (x')^2 - 2x'h - h^2$
$\qquad - 7x' - 7h + 3$

$$y' = (x')^3 + (+3h - 1)(x')^2 + (3h^2 - 2h - 7)x' + h^3 - h^2 - 7x' + 3 - k$$

Setting the $(x')^2$ and $(x')^0$ coordinates equal to zero, we have:

$$+3h - 1 = 0 \qquad h^3 - h^2 - 7h + 3 - k = 0$$

$$h = +\frac{1}{3} \qquad k = h^3 - h^2 - 7h + 3$$

$$k = \left(+\frac{1}{3}\right)^3 - \left(+\frac{1}{3}\right)^2 - 7\left(+\frac{1}{3}\right) + 3$$

$$= +\frac{1}{27} - \frac{1}{9} - \frac{7}{3} + 3$$

$$= \frac{+1 - 3 - 63 + 81}{27} = +\frac{16}{27}$$

$$y' = (x')^3 - \frac{22}{3}x'$$

The coordinates of the center of symmetry, referred to the original axes, are $(h, k) = \left(\frac{1}{3}, \frac{16}{27}\right)$.

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Pupils frequently ask, in connection with the proof of this theorem, why one multiplies each side by q^2 instead of, for example, q^3. The situation has analogies in geometry where one must choose a particular construction or a particular line of attack in order to reach the desired goal. The underscored phrase is the point to be made with the pupil. A line of attack is chosen because it enables us to reach a goal that is known in advance. It works.

If such a specialized line of attack must be remembered, then the proof of the theorem becomes doubtful as a sensible part of the pupil's general education, and we do not recommend that the student be required to reproduce it. He ought to understand it and appreciate its "logical beauty," but not be forced to memorize it.

Page 460. Synthetic Division

A very nice illustration of the process can be arranged as follows: Suppose we divide $a_3x^3 + a_2x^2 + a_1x + a_0$ by $x - m$. By the division identity we have

$$a_3x^3 + a_2x^2 + a_1x + a_0 = (x - m)(b_2x^2 + b_1x + b_0) + R$$
$$= b_2x^3 + (b_1 - mb_2)x^2 + (b_0 - mb_1)x - mb_0 + R$$

Since this equality is an identity, coefficients of like powers of the variables are equal:

$a_3 = b_2$

$a_2 = b_1 - mb_2$ whence $b_1 = a_2 + mb_2$

$a_1 = b_0 - mb_1$ whence $b_0 = a_1 + mb_1$

$a_0 = R - mb_0$ whence $R = a_0 + mb_0$

Now, observe the use of these conclusions in the synthetic division array:

a_3	a_2	a_1	a_0 | m
	mb_2	mb_1	mb_0
a_3	$a_2 + mb_2$	$a_1 + mb_1$	$a_0 + mb_0$

(The procedure outlined above is called for on page 481, problems 6 and 7.)

Page 465. Approximate Solution of Cubic Equations

We recommend, at this stage, the procedure discussed in the text on pages 465-467. However, the teacher may wish to examine other methods. The method called "Horner's Method" is outlined in problem 25 on page 482 of the text.

Newton's method, based on approximations by means of tangents, we present in two forms for the general cubic.

First form. Let r be the best available approximation to a root of the equation $ax^3 + bx^2 + cx + d = 0$. Let $r + k$ be the exact root. Then $f(r + k) = 0 = a(r + k)^3 + b(r + k)^2 + c(r + k) + d$. In the expansion, neglecting the higher powers of k, we have:

$$k = \frac{-(ar^3 + br^2 + cr + d)}{3ar^2 + 2br + c} = -\frac{f(r)}{f'(r)}.$$

Second form. Given the equation $f(x) = 0$ with a an approximate value of a real root and $a + h$ a closer value.

From Taylor's Series we have

$$f(a + h) = f(a) + f'(a)h + f''(a)\frac{h^2}{2} + \cdots$$

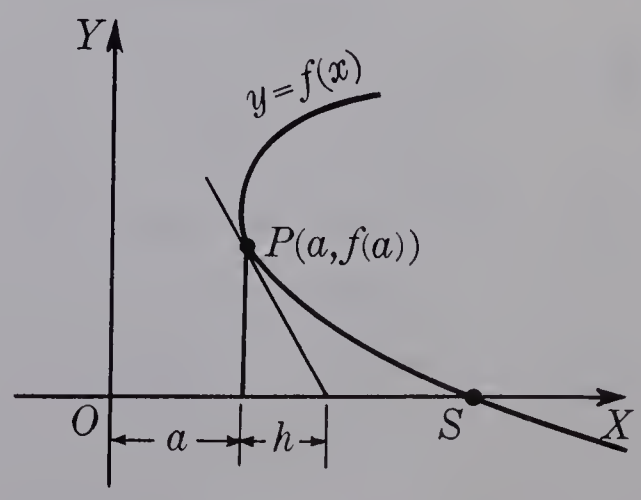
Neglecting higher powers of h, and assuming that $a + h$ is a root so that $f(a + h) = 0$,

$$0 = f(a) + f'(a)h$$

whence $h = -\frac{f(a)}{f'(a)}$

The diagram exhibits the essential point of Newton's method:

$\tan \angle STP = -f'(a) = \frac{f(a)}{h}$. $\therefore h = -\frac{f(a)}{f'(a)}$



Example, using Newton's Method

Find, to the nearest .01, the root between 2 and 3 of the equation

$$2x^3 - 3x^2 - 4x - 5 = 0$$

Solution: $f(x) = 2x^3 - 3x^2 - 4x - 5$

and $f'(x) = 6x^2 - 6x - 4$

Since $f(2) = -9$ and

$f(3) = +10$, we guess that 2.5 may be a reasonable approximation to the desired root.

Then $f(2.5) = -2.5$

$f'(2.5) = 18.5$

$$\therefore h_2 = -\frac{f(2.5)}{f'(2.5)} = -\frac{-2.5}{18.5} = .135$$

Hence 2.5 + h = 2.635 is a second approximation.

$f(2.635) = 0.22112075$

$f'(2.635) = 21.84935$

$$\therefore h_3 = -\frac{f(2.635)}{f'(2.635)} = -\frac{0.22112075}{21.84935} = -0.0101$$

Hence 2.635 + (-0.0101) = 2.6249

$f(2.624) = -0.01767475$

$f(2.625) = +0.00390625$

The process may be continued, using x = 2.625. However, we may conclude that the root, to the nearest .01, is 2.62.

Horner's Method

The following array may help to make clear what is going on at the various steps of Horner's method.

(A)	$x^4 + x^3 - 3x^2 - x - 4 = 0$	Root between 1 and 2
(B)	$x^4 + 5x^3 + 6x^2 - 6 = 0$	Root between 0 and 1
(C)	$x^4 + 50x^3 + 600x^2 - 60{,}000 = 0$	Root between 0 and 10 Root between 7 and 8
(D)	$x^4 + 78x^3 + 1944x^2 + 17122x - 11049 = 0$	Root between 0 and 1
(E)	$x^4 + 780x^3 + 194400x^2 + 17122000x - 110490000 = 0$	Root between 0 and 10 Root between 6 and 7
(F)	$x^4 + 804x^3 + 208656x^2 + 19539904x - 589824 = 0$	Root between 0 and 1

Notes:

Ⓐ We let r be the root.

Ⓑ The roots of Ⓐ are diminished by 1. Hence the root of Ⓑ is given by $r - 1$

Ⓒ Multiply the roots of Ⓑ by 10. Then the root of Ⓒ is $10(r - 1)$

Ⓓ Diminish the roots of Ⓒ by 7. Then the root of Ⓓ is $10(r - 1) - 7$

Ⓔ Multiply the roots of Ⓓ by 10. Then the root of Ⓔ is $10[10(r - 1) - 7]$

Ⓕ Diminish the roots of Ⓔ by 6. Then the root of Ⓕ is $10[10(r - 1) - 7] - 6$

Since $10[10(r - 1) - 7] - 6 = 0.0302$

then $100r - 176 = 0.0302$

$r = 1.760302$

Page 469. Problem 12

The inquisitive may desire a derivation of this formula. Here is one. Volume of dome = volume of spherical cone - volume of cone of revolution

$$= \frac{2}{3}\pi r^2 h - \frac{1}{3}\pi\left(\sqrt{r^2 - (r - h)}\right)^2 (r - h)$$

$$= \frac{2}{3}\pi r^2 h - \frac{1}{3}\pi(2rh - h^2)(r - h)$$

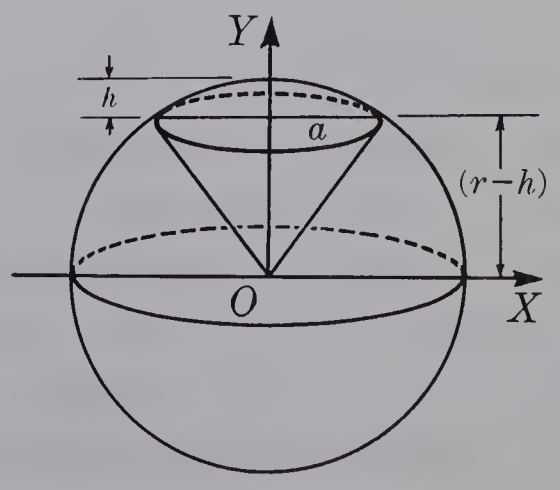
$$= \frac{2}{3}\pi r^2 h - \frac{1}{3}\pi h(2r - h)(r - h)$$

$$= \frac{1}{3}\pi h[2r^2 - (2r^2 - 3rh + h^2)]$$

$$= \frac{1}{3}\pi h(3rh - h^2)$$

$$= \frac{1}{3}\pi h^2(3r - h)$$

$$= \pi h^2\left(r - \frac{1}{3}h\right)$$



$a^2 = r^2 - (r - h)^2$

$a^2 = 2rh - h^2$

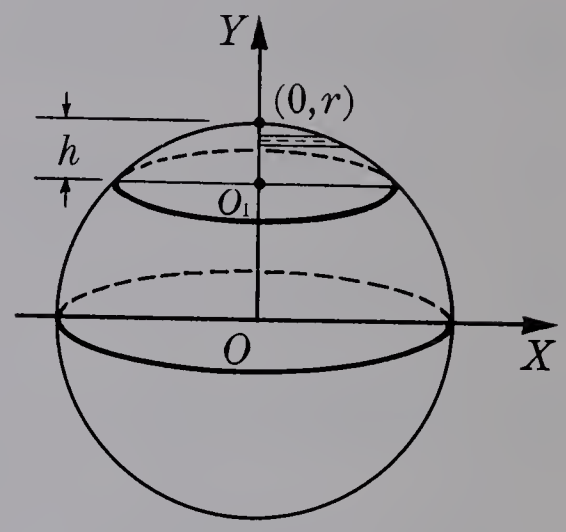
This relationship is another one whose derivation comes more easily from the calculus. Thus:

The equation of circle O may be written as $x^2 + y^2 = r^2$.

The coordinates of O_1 are $(0, r - h)$

$$\triangle V_i = \pi x_i^2 \cdot \triangle y$$

From $x^2 + y^2 = r^2$ we have $x^2 = r^2 - y^2$



$$\therefore V = \pi \int_{r-h}^{r} (r^2 - y^2)dy$$

$$V = \pi \left[r^2 y - \frac{y^3}{3} \right]_{r-h}^{r}$$

$$V = \pi \left[r^3 - \frac{r^3}{3} - \left(r^2 (r - h) - \frac{(r - h)^3}{3} \right) \right]$$

$$V = \pi \left[-\frac{r^3}{3} + r^2 h + \frac{r^3}{3} - r^2 h + r h^2 - \frac{h^3}{3} \right] = \pi h^2 \left(r - \frac{1}{3} h \right)$$

Pages 469-482

The materials on these pages are all labelled as of "B" difficulty. A careful examination of the exercises will show, however, that some of them are not particularly difficult; they are just unnecessary as part of the development of an average course. For example, they would certainly be omitted by a group following Plan B as outlined on page 2 of this manual. It is quite possible to skip all the theoretical developments on these pages and still use the following problems:

pp. 473-474 (B - 1) - 16, 17, 19, 20
pp. 474-476 (B - 2) - 13, 16, 19, 24, 25
p. 478 (B - 1) 1, 2, 3, 4, 5, 6, 7
pp. 478-479 (B - 2) - 1, 2, 3, 4, 5, 6, 7

Page 470. Theorem 1

The so-called Fundamental Theorem of Algebra is the basis for Theorem 1. The Fundamental Theorem of Algebra may be stated as follows: "Every polynomial equation, $f(x) = 0$, of positive degree with complex coefficients has a complex root."

The proof of this theorem is too involved to justify its inclusion in this manual. The interested reader will find proofs in

Birkhoff, G. and MacLane, S., A Survey of Modern Algebra, New York, The Macmillan Co., 1953; pp. 107-109.

Curtiss, D. R., Analytic Functions of a Complex Variable. The second Carus monograph of The Mathematical Association of America, published by The Open Court Publishing Co., Chicago, 1926; pp. 100-101.

Dickson, L. E., First Course in the Theory of Equations. New York, John Wiley and Sons, Inc., 1922; pp. 155-158.

Fine, H. B., A College Algebra. Boston, Ginn and Company, 1904; pp. 588-589.

Page 481

In Exercise 14 and subsequent exercises, there frequently occurs the problem of writing an equation with roots that are related in a particular manner to the roots of a given equation. This problem takes various forms. One form is:

Write an equation with roots 3 less than the roots of

$$x^3 - 2x^2 + x - 1 = 0. \quad (A)$$

Solution: If x_i are the roots of (A), and y_i the roots of the desired equation, then $y_i = x_i - 3$. (B)

Whence $x = y + 3$, and so, substituting $y + 3$ for x in (A) we have

$$(y + 3)^3 - 2(y + 3)^2 + (y + 3) - 1 = 0$$

or $y^3 + 7y^2 + 16y + 11 = 0$ (C)

Each root of (C) is 3 less than the corresponding root of (A). If, in (C), y is replaced by $x - 3$, then the resulting equation, $(x - 3)^3 + 7(x - 3)^2 + 16(x - 3) + 11 = 0$ (D) is identical with (A). The coefficients of (C) can be derived from (D) by dividing (D) by $x - 3$ three times. Since (D) is identical with (A) we should obtain the coefficients of (D) if we divide (A) by $x - 3$. Synthetic division simplifies the process:

$$x^3 - 2x^2 + x - 1$$

1	−	2	+	1	−	1	3
		3		3		12	
1		1		4		(11)	
		3		12			
1		4		(16)			
		3					
(1)		(7)					

Another way of describing the transformation of (A),

$x^3 - 2x^2 + x - 1 = 0$, into Ⓓ,
$(x - 3)^3 + 7(x - 3)^2 + 16(x - 3) + 11 = 0$, is to say that Ⓐ has been expanded in powers of $x - 3$.

This expansion can also be obtained by Taylor's series:

$$f(x) = f(a) + f'(a) \cdot \frac{(x - a)}{\lfloor 1} + f''(a)\frac{(x - a)^2}{\lfloor 2} + f'''(a)\frac{(x - a)^3}{\lfloor 3} + \cdots$$

$f(x) = x^3 - 2x^2 + x - 1$	$f(3) = 11$
$f'(x) = 3x^2 - 4x + 1$	$f'(3) = 16$
$f''(x) = 6x - 4$	$f''(3) = 14$
$f'''(x) = 6$	$f'''(3) = 6$

$$\therefore \quad f(x) = 11 + 16(x - 3) + 7(x - 3)^2 + (x - 3)^3$$
$$\text{or} \quad f(x) = (x - 3)^3 + 7(x - 3)^2 + 16(x - 3) + 11.$$

This chapter can be summarized in some such way as follows.

We have been studying the mathematical objects called polynomials, here denoted by $f(x)$. If we consider all the information that we have thus far accumulated about $f(x)$ we realize that the set of polynomials satisfies the postulates of a field except that division is in general not possible. These postulates are the same ones that the set of integers satisfy and may be stated as below:

Addition	Multiplication
1. $a + b = c$	1. $ab = c$
2. $a + b = b + a$	2. $ab = ba$
3. $a + (b + c) = (a + b) + c$	3. $a(bc) = (ab)c$
4. $a + 0 = a$	4. $a \cdot 1 = a$
5. $a + (-a) = 0$	

$$a(b + c) = ab + ac$$

We say that the set of polynomials, like the set of integers, is a commutative ring with unity. The requirements for a ring do not include the existence of a multiplicative inverse for each non-zero element. Thus the set of polynomials and the set of integers have the same formal structure and are said to be isomorphic.

Actually, our study of polynomials has paid more attention to polynomial equations, $f(x) = 0$, and to polynomial functions, f, defined by the equation, $y = f(x)$, than it has to the formal structure of the set of polynomials in general. We have been interested in the study of facts about $f(x) = 0$ and about $y = f(x)$ because of their usefulness in solving problems and because they are the next

natural step in the study of equations and of functions. The study of the formal structure of the set of polynomials is a step that will be taken in a first course in abstract algebra and has no place at this stage except that the student may be made aware of the possibility. Any book on modern algebra will present a discussion that the teacher may find of use in his own thinking although it would certainly be bad pedagogy to present it to a class at this stage. The teacher is particularly warned that in the formal study of the set of polynomials nothing whatever is assumed about the symbol "x," not even that it is in an integral domain and so "x" is commonly referred to as an indeterminate. This is in marked contrast to the way x is used throughout Chapter 17 where x always is used as a symbol that is replaceable by a number.

Chapter 18. Sequences and Series

The study of sequences and series at this stage can begin in a manner that will interest the student, arouse his curiosity, and excite him a little. We call to the attention of the teacher some points which can serve to unify the content of this chapter.

The first point is that a sequence is a function whose domain is the set of positive integers. This is a new idea at this time, for there was no point in introducing such functions in the introductory remarks about functions in Chapter 9, and each function which has been studied in detail up to now has in general had as its domain the set of real numbers, or at least a continuous subset of the real numbers. By emphasizing the sequence as a function having domain restricted to the positive integers we immediately identify the study of sequences as having points of similarity with the earlier studies of the linear, quadratic, and cubic functions and thus make a smooth entrance into the topic. We are merely studying functions that have the property that their domain is the set of positive integers.

We shall study arithmetic series and observe their association with linear and quadratic functions. In the investigation of geometric series the concept of limit will arise as will the notion of convergence. We shall study the binomial series because of its importance in other branches of mathematics such as calculus and probability.

In the introductory study of sequences and series at this stage we are forced to begin with the simplest aspects of the subject. How is the sequence formed? Can we express the nth term in terms of n? What is the sum of n terms if n is finite? What are some of the puzzle problems that can be contrived out of the elementary definitions and facts? The great question of approximation by use of infinite series is touched on only a little, for its difficulty makes it unsuitable for study at this stage. We do get into the question of convergent and divergent series enough to make the terms meaningful, but hardly enough to make them useful without further study. We think that the establishment of precise criteria for the convergence of series is a subject for study in later courses. We repeat now the three main guide posts for the teacher of this chapter:

1. A sequence is a function whose domain is the set of positive integers. This domain makes the sequence a radically different function from those previously studied.
2. Many problems can be contrived about sequences and series. We concentrate on the problems related to the arithmetic, the geometric, and the binomial series.
3. The convergence of an infinite series gives us a powerful tool for a fruitful study of approximations. We touch on it here but leave most of its development to a later course.

Page 489. Notation

In discussing the notation we suggest that two statements such as $f(x) = x^2 + x$ and $t_n = n^2 + n$ be placed side by side on the blackboard and the similarities and differences in notation and meaning be discussed. Thus we note that $f(x) = x^2 + x$ defines a function continuous on the set of reals; we can find its minimum value; its graph can be drawn without lifting pencil from paper; its domain is $\{x \mid -\infty < x < +\infty\}$, and its range is $\{f(x) \mid f(x) \geqq -\frac{1}{4}\}$. On the other hand, $t_n = n^2 + n$ defines a set of integers that are in one-to-one correspondence with the counting numbers; its graph is a set of discrete points; its domain is $\{n \mid n$ is a positive integer$\}$ and its range is $\{t_n \mid t_n$ is a positive integer given by $t_n = n^2 + n\}$.

Page 496. Sum of an Arithmetic Series

The formula for the sum of an arithmetic series may be found by examining the series of partial sums. Thus if the terms of an arithmetic series are

$$t_1,\quad t_1 + d,\quad t_1 + 2d,\quad t_1 + 3d,\quad t_1 + 4d_1 \cdots$$

then the terms of the series of partial sums are

$$\begin{aligned} S_1 &= t_1 \\ S_2 &= 2t_1 + d \\ S_3 &= 3t_1 + 3d \\ S_4 &= 4t_1 + 6d \\ S_5 &= 5t_1 + 10d \\ S_6 &= 6t_1 + 15d \\ &\vdots \quad\quad \vdots \quad\quad \vdots \end{aligned}$$

To determine S_n we first note the obvious fact that the coefficient of t_1 is n. To express the coefficient of d in terms of n we note that the sequence of coefficients of d is

$$1,\quad 3,\quad 6,\quad 10,\quad 15,\quad 21,\quad \cdots.$$

The first differences of successive terms in this sequence are

$$2,\quad 3,\quad 4,\quad 5,\quad 6,\quad \cdots$$

and the second differences are therefore constant at 1. Hence the coefficient of d must be a quadratic function of n. (See page 485 of Second Course in Algebra with Trigonometry). To determine this quadratic expression, $an^2 + bn + c$, we have

$$\begin{cases} S_1 = 1 = a + b + c \\ S_2 = 3 = 4a + 2b + c \\ S_3 = 6 = 9a + 3b + c \end{cases}$$

The solution of these three equations gives $a = \frac{1}{2}$, $b = -\frac{1}{2}$, $c = 0$. Hence the desired expression is $\frac{1}{2}n^2 - \frac{1}{2}n$. It follows that

$$\begin{aligned} S_n &= nt_1 + d\left[\tfrac{1}{2}n^2 - \tfrac{1}{2}n\right] \\ &= \tfrac{n}{2}[2t_1 + (n - 1)d] \end{aligned}$$

The procedure just outlined for deriving a formula for S_n is not recommended in preference to the method of the text. It is included here as a matter of interest and particularly for those

groups that may have looked at Sidelight 12 on page 484ff. The teacher's attention is also called to the fact that the formula for the sum of a geometric series may be derived in a similar manner.

If the geometric sequence is

$$t_1, \quad rt_1, \quad r^2t_1, \quad r^3t_1, \quad r^4t_1, \quad \cdots$$

then the sequence of partial sums is

$$\begin{aligned} S_1 &= t_1 \\ S_2 &= t_1 + rt_1 \\ S_3 &= t_1 + rt_1 + r^2t_1 \\ S_4 &= t_1 + rt_1 + r^2t_1 + r^3t_1 \\ &\vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \end{aligned}$$

Hence $$S_n = t_1(1 + r + r^2 + r^3 + \cdots + r^{n-1}).$$

The rest of the argument proceeds as on pages 504, 505 of the text.

Page 511 (Top of page)

The proof that the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$ is divergent proceeds as follows.

The sum of the first four terms is

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{2} = 2$$

The sum of the next four terms is

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{4}{8} = \frac{1}{2}$$

The sum of the next eight terms is

$$\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} > \frac{8}{16} = \frac{1}{2};$$

and so on. Thus the sum of the first $4 + 4 + 8 + 16 + \cdots + 2^n = 2^{n+1}$ terms is greater than

$$\begin{aligned} &2 + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} \\ &= \left(\frac{3}{2} + \frac{1}{2}\right) + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} \\ &= \frac{1}{2}(n + 3), \end{aligned}$$

and this increases beyond all limit with n; hence the series diverges to $+\infty$.

Page 525. Proof of the Binomial Theorem for positive integral exponents

In the expansion of $(a + b)^n$ it appears that the coefficient of the term which contains b^r, $(0 \leq r \leq n)$ is $\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$, which may be written $\frac{n!}{r!(n-r)!}$. The symbol $\binom{n}{r}$ is commonly used to represent $\frac{n!}{r!(n-r)!}$. To prove the binomial theorem it is therefore necessary to show that $(a + b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} \cdot b^r$. The general inductive step is then to show that *if* $(a + b)^k = \sum_{r=0}^{k} \binom{k}{r} a^{k-r} b^r$, *then* $(a + b)^{k+1} = \sum_{r=0}^{k+1} \binom{k+1}{r} a^{k+1-r} b^r$.

Now we have

$$(a + b)^{k+1} = (a + b)^k \cdot (a + b)$$

$$= \left(\sum_{r=0}^{k} \binom{k}{r} a^{k-r} b^r \right) (a + b) \text{ if } (a + b)^k =$$

$\sum_{r=0}^{k} \binom{k}{r} a^{k-r} b^r$. The term which contains $a^{k+1-r} b^r$ in this expansion is obtained from $a \cdot \left[\binom{k}{r} a^{k-r} b^r\right] + b \left[\binom{k}{r-1} a^{k-r+1} b^{r-1}\right]$, which gives $\left[\binom{k}{r} + \binom{k}{r-1}\right] \cdot a^{k-r+1} b^r$.

To obtain the required result it is necessary to show that $\binom{k}{r} + \binom{k}{r-1} = \binom{k+1}{r}$.

We have $\binom{k}{r} + \binom{k}{r-1} = \frac{k!}{r!(k-r)!} + \frac{k!}{(r-1)!(k-r+1)!}$

$$= \frac{k!}{(r-1)!(k-r)!}\left[\frac{1}{r} + \frac{1}{k-r+1}\right]$$

$$= \frac{k!}{(r-1)!(k-r)!}\left[\frac{k+1}{r(k-r+1)}\right]$$

$$= \frac{(k+1)!}{r!(k+1-r)!} = \binom{k+1}{r}.$$

Chapter 19. Trigonometric Addition Formulas

One of the objectives of this chapter is to acquaint the student with the definitions of the trigonometric functions as sets of ordered pairs of numbers that are generated without reference to angle measure. This connects the formal study of these functions with the study of functions in general. It makes the derivations on pp. 540ff. completely general, and it prepares the student for the use of the sine and cosine in situations where the concept of angle is irrelevant but the concept of periodic behavior is essential. Examples of the latter occur in the study of simple harmonic motion or the period of a pendulum, phenomena in the general category of wave motion. The student should become comfortable with the notion of a trigonometric function of a number. The brief introduction to the inverses of the sine, cosine and tangent serves to broaden further the student's feeling for functions. These inverses in particular throw revealing light on the importance of domain and range.

A second objective of the chapter is a familiarity with the basic addition formulas and the variants derivable from them. A good way to decide whether or not this objective has been achieved is to demand that the student start with one formula [$\cos(A + B) = \cos A \cos B - \sin A \sin B$] and derive all the others from it. The procedure is simplified, of course, if one starts with the formulas for $\sin(A + B)$, $\cos(A + B)$, $\tan(A + B)$, and derives the others from these three.

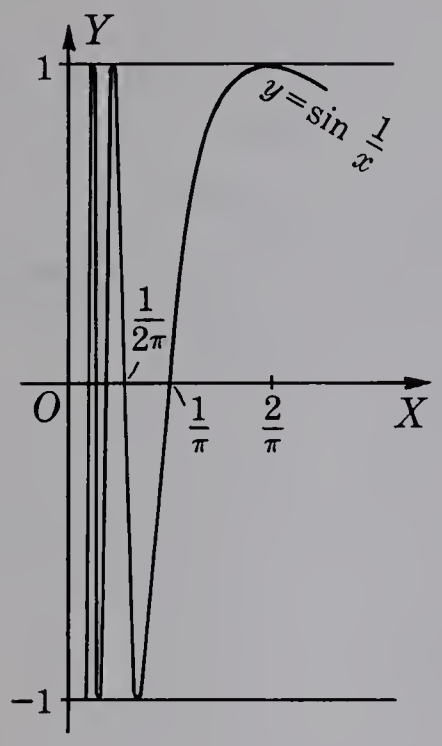
Finally, since the student actually does not need at this stage any concept of the functions that is divorced from the concept of angle, most of the language of the chapter refers to angles, either measured in degrees or in radians. The concept of radian is introduced early and used often because it will generally be used in subsequent study by the student. The semantic jump from "radians" to "real numbers" is largely ignored after page 541 simply because it serves no useful purpose at this time.

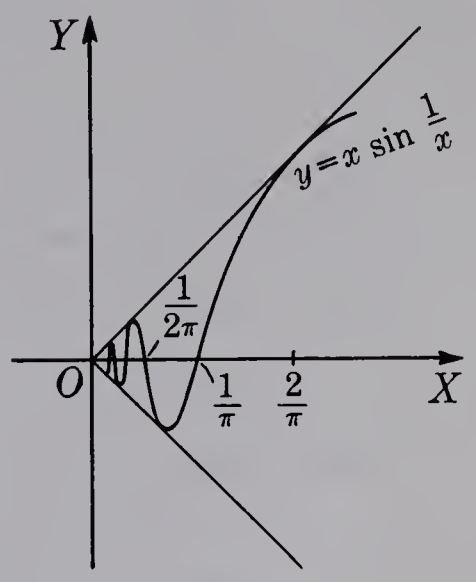
Page 539. Exercises 14, 15, 16

These exercises, of course, are meant to foreshadow the theorem that $\lim_{x \to 0} \frac{\sin x}{x} = 1$. We ask for the geometric argument that leads to this conclusion in Exercise 21 on page 560 of the text. If the interest and ability of the class seem to warrant

it the teacher may wish to introduce the very interesting functions defined by

$$y = \sin \frac{1}{x} \qquad y = x \sin \frac{1}{x}$$





For a brief discussion of these functions see R. Courant, Differential and Integral Calculus, 2d edition, Vol. 1, pp. 53, 54; Interscience Publishers, Inc., New York.

Throughout the remainder of this chapter the reader will find many facts foreshadowed by the exercises. We recommend careful attention to the sequence of the questions, with few or no omissions, so that the full benefit of the build-up may accrue to the student.

Page 540

A class that is good enough to get this far in the book will be interested in the general inquiry mentioned in the first paragraph of this page. A less condensed version may be presented orally in class in some such words as the following.

Let us recall that a function is a set of ordered pairs no two of which have the same first element. If the number of ordered pairs is not finite and we therefore have a rule for associating an element of the range with an element of the domain, then a natural question to raise is this:

If the function has a typical ordered pair of $(x, f(x))$ then how is $f(x_1) + f(x_2)$ related to $f(x_1 + x_2)$?

Example 1. If $f(x) = ax + b$, then

$$f(x_1) = ax_1 + b$$
$$f(x_2) = ax_2 + b$$

$$f(x_1 + x_2) = a(x_1 + x_2) + b$$
$$= ax_1 + ax_2 + b$$
$$\text{or} \quad f(x_1 + x_2) = f(x_1) + f(x_2) - b$$

Example 2. If $f(x) = ax^2 + bx + c$,
then $f(x_1) = ax_1^2 + bx_1 + c$
$f(x_2) = ax_2^2 + bx_2 + c$
$f(x_1 + x_2) = a(x_1 + x_2)^2 + b(x_1 + x_2) + c$
$= ax_1^2 + bx_1 + c + ax_2^2 + bx_2 + 2ax_1x_2$
or $f(x_1 + x_2) = f(x_1) + f(x_2) - c + 2ax_1x_2$

Example 3. If $f(x) = a^x$,
then $f(x_1) = a^{x_1}$
$f(x_2) = a^{x_2}$
$f(x_1 + x_2) = a^{x_1+x_2}$
$= a^{x_1} \cdot a^{x_2}$
or $f(x_1 + x_2) = f(x_1) \cdot f(x_2)$

Example 4. If $f(x) = \log_{10} x$,
then $f(x_1) = \log_{10} x_1$
$f(x_2) = \log_{10} x_2$
$f(x_1 + x_2) = \log_{10}(x_1 + x_2)$
$f(x_1 + x_2) = \log\left(10^{f(x_1)} + 10^{f(x_2)}\right)$

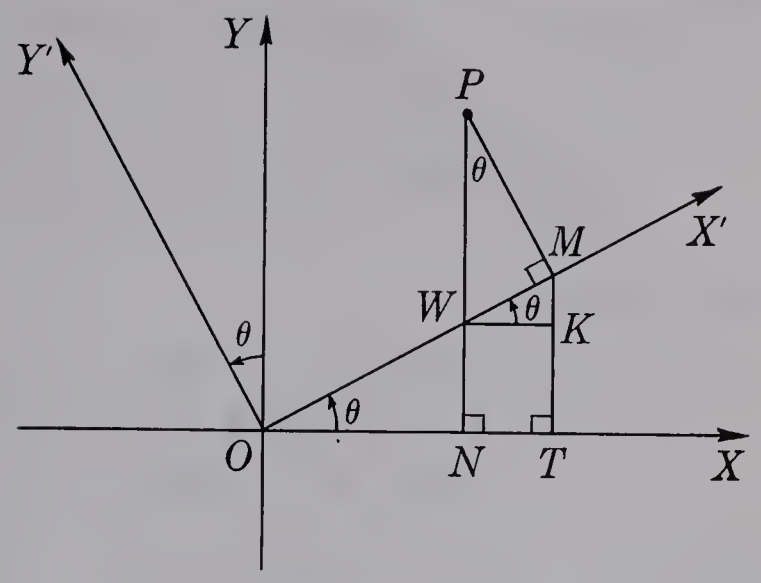
Example 5. If $f(x) = x$
then $f(x_1) = x_1$
$f(x_2) = x_2$
$f(x_1 + x_2) = x_1 + x_2$
or $f(x_1 + x_2) = f(x_1) + f(x_2)$

As can be seen from the above examples, there is no general pattern for the relationship of $f(x_1 + x_2)$ to $f(x_1)$ and $f(x_2)$. It is therefore important to investigate with each new function how $f(x_1 + x_2)$ is related to $f(x_1)$ and $f(x_2)$. The investigation of this relationship for trigonometric functions is the object of our study in this chapter.

Page 559

Some of the exercises on page 559 can be assigned as special projects if the teacher is interested in special projects; e.g., numbers 13 to 16 and the sequence of numbers 17 to 20. The theorem that is presented in Exercise 16(b) is commonly referred to as De Moivre's Theorem.

Double-Angle Formulas Used in Deriving Formulas for Rotation of Axes



If the coordinates of P, referred to the original axes, are x and y, we seek the coordinates of P, referred to the x' and y' axes. We shall call these coordinates x' and y'.

$PM = y' = PW \cos \theta$

$PW = PN - WN$

From $\triangle OWN$, $WN = x \tan \theta$

$PN = y$

$\therefore PW = y - x \tan \theta$ and so

$PM = y' = (y - x \tan \theta) \cos \theta$ or

$y' = y \cos \theta - x \sin \theta$

$MO = x' = OT \sec \theta$ (from $\triangle OTM$)

$OT = ON + NT = x + WK$

$WK = WM \cos \theta$ (from $\triangle WKM$)

$WM = PM \tan \theta = y' \tan \theta$ (from $\triangle WMP$)

$\therefore WK = y' \tan \theta \cos \theta$

$\therefore MO = x' = [x + y' \tan \theta \cos \theta] \sec \theta$

$= x \sec \theta + (y \cos \theta - x \sin \theta) \tan \theta \cos \theta \sec \theta$

$$x' = \frac{x}{\cos \theta} + y \sin \theta - \frac{x \sin^2 \theta}{\cos \theta}$$

$$= x\left(\frac{1 - \sin^2 \theta}{\cos \theta}\right) + y \sin \theta$$

$x' = x \cos \theta + y \sin \theta$

To find formulas for x and y in terms of x', y' and θ, we solve the equations

$$\begin{cases} x' = x \cos \theta + y \sin \theta \\ y' = y \cos \theta - x \sin \theta \end{cases}$$

for x and y, obtaining:

$$\begin{cases} x = x' \cos \theta - y' \sin \theta \\ y = x' \sin \theta + y' \cos \theta \end{cases}$$

Example: Given: $x^2 + 2xy + 3y^2 = 8$.

Required: Rotate the axes so that the xy-term will become zero.

Since $\begin{cases} x = x' \cos\theta - y' \sin\theta \\ y = x' \sin\theta + y' \cos\theta \end{cases}$

we have $(x' \cos\theta - y' \sin\theta)^2 +$

$2(x' \cos\theta - y' \sin\theta)(x' \sin\theta + y' \cos\theta) +$

$3(x' \sin\theta + y' \cos\theta)^2 = 8$

This becomes

$$(x')^2 [\cos^2\theta + 2 \sin\theta \cos\theta + 3 \sin^2\theta] +$$
$$x'y'[4 \sin\theta \cos\theta - 2 \sin^2\theta + 2 \cos^2\theta] +$$
$$(y')^2[\sin^2\theta - 2 \sin\theta \cos\theta + 3 \cos^2\theta] = 8$$

Since the coefficient of $x'y'$ is to be zero, we have

$$4 \sin\theta \cos\theta - 2 \sin^2\theta + 2 \cos^2\theta = 0$$
$$2(\sin 2\theta + \cos 2\theta) = 0$$
$$\frac{\sin 2\theta}{\cos 2\theta} + \frac{\cos 2\theta}{\cos 2\theta} = 0$$
$$\tan 2\theta + 1 = 0$$

whence

$$\tan 2\theta = -1$$
$$2\theta = 135°, 315°$$
$$\theta = 67\tfrac{1}{2}°, 157\tfrac{1}{2}°$$
$$\sin 2\theta = \tfrac{1}{2}\sqrt{2}$$
$$\cos 2\theta = -\tfrac{1}{2}\sqrt{2}$$

$$(x')^2 [\sin 2\theta + 1 + (1 - \cos 2\theta)] +$$
$$(y')^2 [1 + 1 + \cos 2\theta - \sin 2\theta] = 8$$

whence $(x')^2 + (3 - \sqrt{2})(y')^2 = 4(2 - \sqrt{2})$

We thus see that the given equation is the equation of an ellipse.

ANSWERS TO TESTS IN THE TEXT

Page 11

1. a. 15.6 b. 752 c. 690 d. 270

2. a. F b. T c. F d. T e. F f. F g. T h. F
i. T j. T k. T l. F m. T n. T o. F p. T q. T

3. $-3\frac{1}{2}$

4. $6n + 15$

5. a. The theorem: If $ac = bc$ and $c \neq 0$, then $a = b$.
   b. The theorem: If $a = b$, then $ac = bc$.
   c. The theorem: If $ac = bc$ and $c \neq 0$, then $a = b$.
   d. The distributive axiom.
   e. The theorem: If $a = b$, then $a + c = b + c$.
   f. The theorem: If $ac = bc$ and $c \neq 0$, then $a = b$.

Page 24

1. $x = -6$ 2. $x = 6$ 3. $x = 2\frac{1}{2}$ 4. $x = 1\frac{2}{3}$ 5. $x = -\frac{1}{3}$

6. $x = 3000$ 7. $x = 6$ 8. $x = 2\frac{1}{5}$ 9. $\{5, -9\}$ 10. $\{3, \frac{1}{3}\}$

11. 3 12. $\frac{1}{6}$ 13. $\frac{7}{8}$ 14. $x > 5$ 15. $x < -\frac{1}{2}$ 16. $x > -1\frac{1}{2}$

17. $\{-7 < x < 3\}$ 18. $\{x < -2\} \cup \{x > 5\}$

Page 46

1. $x = 2$ 2. $(x, y) = (-\frac{1}{3}, \frac{1}{6})$ 3. 1 4. $(x, y, z) =$ $(9, -2, 3)$ 5. \$12,000 6. $1:37\frac{1}{2}$ P.M. 7. $6\frac{2}{3}$ 8. $t = \frac{1}{6}$

9. $a = 0.02$, $b = 1.3$

Page 71

1. $3x(x + 3)(x - 3)$
2. $(2n - 3)(n + 1)$
3. $(3x + 4)(x - 3)$
4. $3(x - 4)(x + 1)$
5. $(x^2 + 1)(x + 1)(x - 1)$
6. $x(x + 1)(x^2 - x + 1)$
7. $(x^2 - 3)^2$
8. $x^2(2x + 3)(x - 2)$
9. $3n(2 + n)(2 - n)$
10. $6(3 + x)(1 - x)$
11. $(t - 2)(3a + 2)$
12. $(3t + x - 2y)(3t - x + 2y)$
13. $(n - 1)(3x - 1)$
14. $(x + y)(x - y - 4)$
15. $(x - 2y)(x^2 + 2xy + 4y^2)$
16. $5(2t - 5)(t + 1)$
17. $(a + b + c)(a + b - c)$
18. $(a + b - c)(a - b + c)$
19. $(x + 1)(x - 1)(x^2 + x + 1)$
20. $2(y - 2)(4y - 1)$
21. $(4 + 3nt)(3 - 4nt)$
22. $2(x + 4)(x + 1)(x - 1)$

23. $(x - y)(2x - 2y - 3)$
24. $-5(x - y)(x - 2y)$
25. $n^2(n + 4)$
26. $x = 4\frac{2}{3}$
27. $x = 2\frac{1}{2}$
28. $x = -\frac{1}{5}$
29. $64x^3 - 48x^2y + 12xy^2 - y^3$

Page 96

1. a. $\frac{4x}{4y}$ b. $\frac{2x}{4y}$ c. $\frac{y}{4y}$ 2. $0 < n < 1$ 3. $\frac{p}{p - 1}$

4. a. $x = 2$ b. $x = 2000$ c. -1999 d. The value of the fraction approaches 1.

5. $\frac{ax}{(x + a)(x - a)}$ 6. $\frac{1 + y}{y}$ 7. $-\frac{1}{3}$ 8. $\frac{3x}{2}$
9. $\frac{1}{2}$ 10. $\frac{1}{m - 1}$

Page 113

1. $\{-\frac{1}{2}, \frac{5}{3}\}$ 2. $\{\frac{3}{2}, -5\}$ 3. $\{15, -8\}$ 4. $\{0, 2\frac{1}{2}, -2\}$
5. $\{10, 24\}$ 6. $\{-24, 6\}$ 7. $x = 3$ 8. $\{-7, 3\}$ 9. -12, -11 or $13, 14$ 10. 9 in. and 24 in. 11. 135 12. $x = 100$

Page 130

1. 4.5 (in.) 2. $31\frac{1}{2}$ (in.) 3. $x : y = 5 : 4$ 4. $\frac{3}{4}$
5. 24 (in.) and 16 (in.) 6. a. $n = \frac{2c}{4 - c}$ b. $n = 398$
7. $A = 9x - 9$ 8. $x : z = 10 : 3$ 9. $y_2 = \frac{27}{125}y_1$ 10. a. $\frac{3}{8}n$
b. $5 : 11$

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1. 0 2. a 3. $\sqrt[3]{3}$ 4. $\frac{3}{4}\sqrt{2}$ 5. 8 6. $\frac{1}{3}\sqrt{3}$ 7. $\frac{5}{6}\sqrt{6}$
8. $\sqrt{2}$ 9. $\sqrt{2} + 1$ 10. $\frac{1}{3}$ 11. $5 + 2\sqrt{6}$ 12. -5
13. $\frac{1}{2}(3 + \sqrt{3})$ 14. $\frac{7 + 3\sqrt{5}}{2}$ 15. $17 - 12\sqrt{2}$ 16. 1
17. $\frac{1}{3}(-1 - \sqrt{7})$ 18. $\{1 + \frac{1}{3}\sqrt{6}, 1 - \frac{1}{3}\sqrt{6}\}$ Sum of roots = 2. Product of roots = $\frac{1}{3}$. 19. 0.79, -2.12 20. $2(3 + \sqrt{7})$
21. a. 2 b. $2x + 2 - 2\sqrt{x} \cdot \sqrt{x + 2}$ 22. no solutions

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1. y varies directly as t. 2. 9 : 4 3. The set of real numbers, excluding 3 4. $y = 2x + 1$ 5. $y = 12.5$ 6. 1000 : 1

7. $y = 4.5$ 8. 2.5; $x > 2.5$ 9. The value of y is halved.

10. a. $\{x > 2\} \cup \{x < 0\}$ b. $\{0 < x < 2\}$ c. $\{0, 2\}$

11. $y = 6.0$ 12. $(6, -3)$ 13. $y = \frac{1}{2}(17 - 5t)$; $t = -\frac{3}{5}$

14. 72.8% 15. $F = \frac{25v^2}{2r}$

Page 247

1. $\frac{4}{9}$ 2. $\frac{5}{x^2}$ 3. a 4. $-\frac{1}{3}$ 5. $\frac{4}{729}$ 6. 2^{5a} 7. m^{5a}

8. 4 9. $\frac{1}{2}$ 10. 25 11. a. 4 b. $\frac{1}{8}$ c. $-\frac{1}{64}$ 12. a. 1.6

b. 0.3 13. 0.6584 14. V = 405.9 15. $x = \frac{\sqrt{a}}{b}$ 16. a. $\frac{2}{3}$

b. $-\frac{2}{3}$ 17. 2 18. $\frac{3}{2}$ 19. 2.183 20. -1.513

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1. a = 6.319 (in.) c = 14.01 (in.) 2. 36°52', 53°8', 90°

3. B = 48°11' 4. X = 57°2' 5. 53.3 (ft.) 6. 98°16'

7. 53.99% or about 54% 8. 50°26'

Pages 296-297

1. k = 6 2. slope $= \frac{3}{4}$; y-intercept = -3. 3. a. 8 b. $\frac{3}{4}$

4. $x + 2y + 1 = 0$ 5. $y = x + 1$ 6. yes 7. $x - 3y - 11 = 0$

8. $2x - 3y - 10 = 0$ 9. k = 2 10. a. k = 4 b. $k = -\frac{1}{3}$

11. D = (4, 1) 12. $2x - 3y + 7 = 0$ 13. A = 39 sq. units

14. $\frac{\triangle y}{\triangle x} = -\frac{5}{2}$; y is decreased by 12 when x is increased by 4.8.

15. $32 - 0.5x$ is decreased by 5 when x is increased by 10.

16. $f(x) = 3x - 8$

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1. $x = \{1 + \frac{i\sqrt{3}}{3}, 1 - \frac{i\sqrt{3}}{3}\}$ 2. a. $k = -3\frac{1}{8}$ b. $k > -3\frac{1}{8}$

3. $x = \{1.98, -0.78\}$ 4. $112\frac{1}{2}$ 5. k = -5 6. k = 19; maximum value = 19. 7. $f(x) = \frac{1}{2}x^2 - 3x + 4$ 8. $2x^2 - 2x + 1 = 0$

9. a. 2 b. $\frac{1}{2} - \frac{1}{2}i$ 10. -25; (5, -20). 11. $f(x) = 2x^2 - 2x - 4$

12. $k = \pm\frac{3}{4}$

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2. The line is tangent to the circle.

3. a. $(x - 6)^2 + (y + 2)^2 = 9$; $x^2 + y^2 - 12x + 4y + 31 = 0$

b. $(x + 2)^2 + (y - 3)^2 = 25$; $x^2 + y^2 + 4x - 6y - 12 = 0$

4. a. (0, 3); 3 units. b. $(\frac{1}{2}, -2)$; $1\frac{1}{2}$ units. 5. (-1, 2), (-2, 1) 6. $y = x + 3$ 7. $(\frac{18}{5}, \frac{8}{5})$, (0, -2)

8. Domain: $|x| \geqq 2$ Range: $y \geqq 0$

9. a. Rectangular hyperbola; Asymptotes: $y = 0$, $x = 0$.

b. Hyperbola; y-intercepts: 2, -2. Asymptotes: $y = \pm \frac{1}{\sqrt{6}}x$

c. Parabola; x-intercepts: 0, 2.

d. Ellipse; x-intercepts: 5, -5. y-intercepts: $\sqrt{10}$, $-\sqrt{10}$.

10. a. $\{(2, -1), (-2, -1), (\frac{3}{5}\sqrt{5}, \frac{6}{5}), (-\frac{3}{5}\sqrt{5}, \frac{6}{5})\}$

b. $\{(6, -4), (-6, 4), (4, -6), (-4, 6)\}$

11. $(x - 4)^2 + (y - 5)^2 = 45$ 12. $m = \pm \frac{4}{3}$

Page 425

1. 134°26', 225°34' 2. a. $\cos \theta = -\frac{\sqrt{5}}{3}$ b. $\sin \theta = \frac{\sqrt{3}}{2}$ or $-\frac{\sqrt{3}}{2}$ 3. b = 15.65, c = 19.60 4. $\cos B = \frac{11}{16}$

5. $200\sqrt{2}$ sq.in. 6. $\angle B = 53°8'$ or $126°52'$ 7. 120° 8. 53°8' or 126°52' 9. 10.6 in. 10. a. $\angle ABC = 109°28'$

b. BD = 6.40 in.

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1. a. $\frac{\sqrt{3}}{2}$ b. $-\sqrt{3}$ c. -1 2. a. $-\frac{1}{2}$ b. 2 c. 0

3. $\cos \theta$ 4. $\sin A$ 6. $x = \{56°19', 236°19'\}$

7. $x = \{60°, 300°\}$ 8. $\theta = \{210°, 330°, 270°\}$ 9. (b)

Page 483

1. $x = \{-1, 1, i\sqrt{2}, -i\sqrt{2}\}$ 2. k = -19 3. 4

4. $(x - 1)(x - 4)(x + 5)$ 6. Quotient $= 2x^3 + x^2 - 2x - 1$ Remainder = 9 7. $\{2, \frac{1}{2} + \frac{1}{2}i\sqrt{3}, \frac{1}{2} - \frac{1}{2}i\sqrt{3}\}$ 8. $\{x < -1\} \cup \{0 < x < 2\}$ 9. No; $x = \{2, -1 + i\sqrt{2}, -1 - i\sqrt{2}\}$ 10. -10

11. 2; no. 12. 2.28

Page 531

1. $t_{10} = 58$; $S_{10} = 310$. 2. $t_8 = \frac{32}{81}$ 3. a. $x = 6$
b. $x = 4\frac{1}{2}$ 4. 10 5. $t_9 = 3$ 6. 3.6 7. $S_8 = 31\frac{7}{8}$
8. $S_{12} = 426$ 9. $S_n = 2^n - 1$ 10. $S_n = n(n + 1)$ 11. \$676
12. $1 + 10x + 45x^2 + 120x^3 + \cdots$

Page 561

1. a. $\frac{5\pi}{6}$ b. $\frac{7\pi}{4}$ 2. a. -1 b. $-\frac{1}{2}$ c. 1 3. $\sin 2x$
4. $\cot \theta$ 5. $\tan \theta = \frac{1}{5}$ 6. a. $\frac{1}{2}$ b. $\frac{1}{4}$ 7. $\frac{4}{3}$ 8. 2
9. $\cos \frac{1}{2}\theta = \frac{1}{3}\sqrt{6}$ 10. $\tan \theta = \{\frac{1}{2}, -2\}$ 11. $0 \leq x \leq \frac{\pi}{4}$
12. $x = \{90°, 210°, 270°, 330°\}$ 13. $\tan x$ 14. $x = \{120°, 240°\}$
15. $\theta = \tan^{-1}\sqrt{k}$

ANSWERS TO SEPARATE ACHIEVEMENT TESTS

Test 1. 1. a. {-6} b. No solution c. All values of x
2. a. If $a + c = b + c$, then $a = b$. b. If $ac = bc$ and $c \neq 0$, then $a = b$. 3. a. -6 b. $\frac{4}{3}$ 4. a. $y = 6t - 4$ b. $y = -7$
5. a. $x = 2y$ b. $x \geq 0$ 6. $x = 3\frac{1}{5}$ 7. -4 8. $x = \frac{8}{9}$
9. a. {8, -4} b. $-4 < x < 8$ 10. a. $x \geq 4$ b. $x > 10$

Test 2. 1. (2, 7) 2. $-\frac{1}{3}$ 3. $(-1, 2\frac{1}{2})$ 4. $(\frac{1}{3}, -\frac{1}{2})$
5. $\frac{3}{5}$ 6. $3\frac{1}{2}$ m.p.h. 7. $y = 1000 - 0.015x$ 8. 12
9. 168, or 14 doz.

Test 3. 1. a. $5x$ b. $\frac{5}{6}x$ c. 13 2. $-\frac{1}{4}$ 3. a. $2x - 1$
b. $2x + 7$ 4. -4 is a solution. 5. $x = -2\frac{2}{3}$ 6. $6n^2 + 2$
7. $4a^2 - 2a + 1$ 8. a. $5(x + 3)(x - 3)$ b. $(2n + 3)^2$
c. $(9 + x)(8 - x)$ d. $(3x + 2y)(x - 5y)$ e. $(x - y)(x^2 + xy + y^2)$
f. $(x - 2a)(x - 3)$ g. $(x + 3)(x + 2)(x - 2)$
h. $(x + a + 2b)(x + a - 2b)$ i. $(2a + y - 3)(2a - y + 3)$
j. $(x + 4)(x + 3)(x - 2)$

Test 4. 1. $\frac{2 - 2x}{2 - x}$ 2. 4.2 3. $\frac{ab}{a + b}$ 4. $\frac{k + 1}{k + 2}$
5. $-3\frac{3}{8}$ 6. $x + 2y - 1$ 7. $-\frac{3}{2}$ 8. $\frac{x}{x - 1}$ 9. $0 < x < 4\frac{1}{2}$
10. 1 11. $\frac{1}{2}$ 12. $x + 1$

Test 5. 1. $x < 5\frac{1}{3}$ 2. {0, 5} 3. $\{-2, 3\frac{1}{2}\}$ 4. $-\frac{1}{3}$
5. $\{\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\}$ 6. a. $(3x - 11)(2x + 7)$ b. $5(2 - x)(4 + 2x + x^2)$
c. $(x + 2a + b)(x + 2a - b)$ d. $(x + 4)(x + 5)(x - 1)$
7. $\frac{2}{x - 2}$ 8. $x = \frac{7}{13}$ 9. 30', 40' or 20', 60' 10. 30

Test 6. 1. $\frac{5}{2}$ 2. $\frac{6}{7}$ 3. 18.75 in. 4. $x = \frac{2ab}{b - 3a}$
5. 48 in. 6. $x = \frac{3ab}{a + b}$ 7. a. $n = \frac{4s}{s - 3}$ b. 28
8. $\{-b, \frac{1}{2}(a + b)\}$ 9. 8 10. 1

Test 7. 1. Addition, multiplication, subtraction. 2. Addition 4. $n - 4$ 5. $\frac{n}{n + 2}$ 6. $3x + 13$ 7. $x = 3$
8. $\{-2, -1, 0, 1, 2\}$ 9. $\frac{4}{9}$ 10. $x = 1$ 11. (6, 2) 12. $\frac{10}{3}$
13. 51 cents 14. $(2x - 3)(x - 2)(x + 2)$
15. $\{\frac{1}{2}(a + 3b), \frac{1}{2}(a - 3b)\}$ 16. 2 17. $\frac{2x}{2x - 1}$ 18. $n = \frac{8A}{A + 3}$
19. $\{\frac{3}{2}, -\frac{1}{2}\}$ 20. 400

Test 8. 1. a. 1167 b. 491.2 c. 0.02491 2. 33.4 ft. 3. a. $\sqrt{3}$ b. $\sqrt{3} + \sqrt{2}$ c. $2(\sqrt{5} + 1)$ 4. a. $|x| \leq 3$ b. $\frac{5}{2}$ 5. a. $\frac{3}{2}$ b. $\sqrt{2}$ c. $\frac{3}{2}\sqrt[3]{2}$ d. $\frac{17}{6}\sqrt{3}$ 6. a. $-\frac{7}{8}$ b. $2\sqrt{10}$ 7. $\{3 + \sqrt{5}, 3 - \sqrt{5}\}$ 8. $x = 2 + \frac{1}{2}\sqrt{10}$ 9. $\{2.12, 0.28\}$ 10. $x = -\frac{3}{4}$; 1 is an extraneous value.

Test 9. 1. There are two values of y for each positive value of x. 2. $y = x + 2$ 3. Straight line, y-intercept 10, x-intercept 5. 4. $x > 2.5$ 5. $x < 1$ 6. a. $y = t$ b. 3 7. a. $x \geq 2$ b. $y \geq 0$ 8. $y = 2.5x$ 9. 2.4 10. $\frac{8}{27}$ 11. 3.6 12. Increase of 100%

Test 10. 1. a. 3 b. 16 c. $\frac{8}{9}$ 2. a. $\frac{2}{3}$ b. $\frac{3}{2}$ c. $\frac{1}{9}$ 3. a. 4 b. 4 is a solution. 4. a. $\frac{1}{2}(x - 1)$ b. $18a^3$ 5. 0.007558 6. 0.2312 7. a. 4 b. $12\frac{1}{2}$ 8. a. -2 b. $\log c = \frac{1}{2}(\log a - 3 \log b)$ 9. a. $\log r = \frac{1}{2}(\log A - \log \pi)$ b. 4.324 in. 10. 2.35

Test 11. 1. $\{3, -\frac{1}{2}\}$ 2. $(3x - 5)(x + 1)(x - 1)$ 3. $x = \frac{ab}{a + b}$ 4. $-\frac{x}{x + a}$ 5. $2(3 + \sqrt{5})$ 6. $3\frac{7}{8}$ 7. 30 8. a. Straight line, y-intercept 6, x-intercept 4. b. $x \leq 4$ 9. $2 < x < 8$ 10. $\{1 + \frac{1}{2}\sqrt{2}, 1 - \frac{1}{2}\sqrt{2}\}$ 11. 5 12. $\frac{4}{a}$ 13. 0.5741 14. $x = 0$; 4 is extraneous. 15. a. 100 b. $a = \frac{b^2}{\sqrt[3]{c}}$

Test 12. 1. a. 2 b. $\frac{1}{5}\sqrt{5}$ c. $\frac{1}{2}\sqrt{5}$ 2. a. AC = n cot x° in. b. AB = n csc x° in. 3. 54°28' 4. 5.82 in. 5. 10 ft. 2 in. 6. 73°44' 7. 39°24' 8. 56.02 in.

Test 13. 1. a. $4\sqrt{5}$ units b. -2 c. (1, 6) 2. $\{14, -10\}$ 3. 3 4. a. (0, 4) b. (-8, 0) 5. a. 5 b. Straight line, y-intercept 2, x-intercept $3\frac{1}{3}$. 6. $3x - y - 4 = 0$ 7. $\{2, 7\}$ 8. $x + 3y + 1 = 0$ 9. $x + 3y = 0$ 10. (19, 17)

Test 14. 1. -2 2. Concurrent at (3, 4) 3. $x + 2y + 3 = 0$ 4. $\Delta x = -8$ 5. $y = 3.5x - 8$ 6. 5 7. a. f(x) is decreased 15. b. $x \leq 4$ 8. a + 2b 9. $f(x) = 2x + 4$ 10. 4

Test 15. 1. a. $-\frac{1}{3}$ b. The set of reals excepting 2. 2. $(2x + y + 1)(2x - y - 1)$ 3. $\{\frac{2}{3}, 1, -1\}$ 4. $\frac{3}{4}$ 5. $\frac{4}{3}$

6. $-\frac{a+2}{a}$ 7. $\{\frac{3}{2}, -3\}$ 8. $x > 5$ 9. $3x - y - 4 = 0$
10. $\frac{1}{2}(5 + \sqrt{5})$ 11. 0.5144 12. 11.10 in. 13. y is increased 15.
14. $\frac{4x^3}{9}$ 15. 2 16. $f(t) = 2t + 6$ 17. 16
18. $\{2 + 2\sqrt{3}, 2 - 2\sqrt{3}\}$ 19. $2x - 3y + 5 = 0$

Test 16. 1. a. $\{2, -\frac{4}{3}\}$ b. $\{1 + \frac{1}{2}i\sqrt{2}, 1 - \frac{1}{2}i\sqrt{2}\}$
2. a. Parabola, x-intercepts 4 and -1, vertex $(\frac{3}{2}, \frac{25}{4})$.
b. Parabola, tangent at (2, 0) to x-axis, y-intercept 12.
3. a. $\{-1, \frac{5}{2}\}$ b. $\{-1 < x < \frac{5}{2}\}$ 4. 4 5. a. $x < -2$
b. $\{x < -1\} \cup \{x > 6\}$ 6. a. $5 - 2i\sqrt{2}$ b. $\frac{1}{5} + \frac{2}{5}i$
7. a. $x = 3$, $y = -4$ b. 2 9. a. 16 b. Domain is $\{x|x$ is a real number$\}$; range is $\{y|y \leqq 16\}$. 10. $(-\frac{5}{4}, -\frac{25}{8})$

Test 17. 1. a. $\{-2 \leqq x \leqq \frac{7}{2}\}$ b. $-15\frac{1}{8}$ 2. D = -8, hence there are no real zeros. 3. a. $1 - 5i$ b. -6 4. Tangent at $(\frac{3}{2}, 0)$ to x-axis, y-intercept 4.5. 5. a. $\frac{16}{3}$ b. 8 c. 4
6. $2x^2 - 2x + 3 = 0$ 7. $f(t) = 2(3 - t)(1 + t)$ 8. 1250 sq.ft.
9. $\{k < 0\} \cup \{k > 3\}$ 10. $PF = \frac{k^2}{4a} + a$

Test 18. 1. Center (5, -4); radius 3 2. $x^2 + y^2 - 8x + 6y = 0$ 3. $x^2 + y^2 + 4x = 0$ 4. The graphs are: a circle with center (0, 0) and radius 5, and a vertical line $x = 3$. The region is that portion of the circle and its interior which lies to the right of the line, including that segment of the line with endpoints (3, 4) and (3, -4). 5. $\{(2, -1), (8, -10)\}$
6. a. $|x| \leqq 2$ b. $0 \leqq y \leqq 1$; Graph is one-half an ellipse with center (0, 0), x-intercepts +2 and -2, and y-intercept 1.
7. a. Ellipse, x-intercepts +2 and -2, y-intercepts +3 and -3.
b. Hyperbola, x-intercepts +2 and -2. c. Two lines through (0, 0), slopes $+\frac{1}{2}$ and $-\frac{1}{2}$. d. Circle, center (0, 0), radius 2.
8. a. $2x - y - 4 = 0$ b. (3, 2), $(\frac{1}{5}, -3\frac{3}{5})$ c. $\frac{14}{5}\sqrt{5}$ units

Test 19. 1. a. Third or fourth b. First or fourth
c. Second 2. a. 1 or -1 b. $-\frac{1}{4}\sqrt{7}$ 3. a. 270° b. 135°, 225°
4. a. $\frac{\sqrt{3}}{2}$ b. $-\frac{\sqrt{2}}{2}$ c. 1 d. $-\frac{1}{2}$ e. 0 f. $\frac{1}{2}$
5. $\{0° \leqq \theta \leqq 60°\} \cup \{300° \leqq \theta \leqq 360°\}$ 6. 111°48' 7. 41°49' or 138°11' 8. $\frac{1}{3}\sqrt{6}$ 9. 124.5 ft. 10. $\sqrt{76}$ in. and 14 in.

Test 20. 1. a. $-\frac{2}{3}\sqrt{3}$ b. -1 c. 2 d. 0 2. $2\frac{1}{2}$
3. a. $-\frac{1}{5}\sqrt{5}$ b. 8 4. 150° is a solution. 5. {123°41', 303°41'}
6. {90°, 270°} 7. a. 1 b. csc A 8. tan θ
9. {68°12', 248°12', 135°, 315°}

Test 21. 1. Parabola; (3, 0), (1, 0), (0, 3); vertex (2, -1). 2. $\{0 < t < 5\}$ 3. 5 4. $-1 + i\sqrt{2}$ 5. Center $(\frac{1}{2}, -1)$, radius $\frac{3}{2}$ 6. a. +2, -2 b. y-intercepts +2 and -2.
7. $(4, \frac{3}{2})$ and (-3, -2) 8. a. $\frac{2}{3}\sqrt{10}, -\frac{2}{3}\sqrt{10}$ b. $-1\frac{1}{4}$ 9. 2 sin θ
10. {90°, 270°, 26°34', 206°34'} 11. 17 in.; $\frac{8}{17}$; 84 sq.in.

Test 22. 1. $\{\frac{3}{2}, -\frac{3}{4} + \frac{3}{4}i\sqrt{3}, -\frac{3}{4} - \frac{3}{4}i\sqrt{3}\}$ 2. $x(x - 2)^2$
3. a. $\sqrt{5}, -\sqrt{5}$ b. 2 4. a. -1 b. 7 c. $2x^2 - x - 4$; 6.
5. $(x - 1)(x + 2)(2x - 1)$ 6. a. $\{x > -\frac{4}{3}\}$ b. $\{-\frac{4}{3} < x < 2\}$
c. $\{-\frac{4}{3} < x < 0\} \cup \{x > 2\}$ 7. b. $\{\frac{2}{3}, -\frac{1}{2} + \frac{1}{2}i\sqrt{5}, -\frac{1}{2} - \frac{1}{2}i\sqrt{5}\}$
8. $\{0 \leqq x \leqq 2\} \cup \{x \leqq -2\}$ 9. Between 2 and 3; 2.4
10. 2.66 in.

Test 23. 1. a. $(x + 1)^2(2x - 1)$ b. $x > \frac{1}{2}$ 2. $\{-\frac{1}{2}, -\frac{1}{2}, i, -i\}$
3. $2 - i\sqrt{3}, -4$ 4. $3x^3 + 8x^2 + 6x + 4 = 0$ 5. $f(t) =$
$2t(t + 1)(t - 2)$ 6. $\frac{3}{4}$ 7. $\{-1 \leqq x \leqq 0\} \cup \{x \geqq 1\}$
8. $F(x) = \frac{-x(x - 4)^2}{4}$ 9. -27 10. $p = 1 - q^2$

Test 24. 1. $t_n = 3n + 5$ 2. $S_{12} = -156$ 3. $S_{17} = 408$
4. a. $S_n = 12n - 2n^2$ b. 20 5. $r = -\frac{2}{3}$, $t_1 = 81$
6. a. $|r| < 1$ b. $2\frac{2}{5}$ 7. $S_{10} = 95\frac{29}{32}$ 8. $\{1, \frac{5}{2}\}$
9. a. $\$7500(0.8)^8$ b. \$1260 10. a. $a^5 + 5a^4b + 10a^3b^2 +$
$10a^2b^3 + 5ab^4 + b^5$ b. $1 - 16x + 112x^2 - 448x^3$ c. 1.194

Test 25. 1. $312\frac{1}{2}$ 2. 2 + i 3. (-1, -6) 4. -5
5. $F(t) = 2(t - 3)(2t + 1)$ 6. $D < 0$ 7. a. $-\frac{1}{2}\sqrt{3}$ b. -1
c. $\sqrt{2}$ 8. $\left\{\frac{p}{\sqrt{q^2 - p^2}}, \frac{-p}{\sqrt{q^2 - p^2}}\right\}$ 9. $4\sqrt{6}$ in.
10. $2r^2\sqrt{2}$ sq.in. 11. 120° 12. 60°, 240° 13. cos A
14. 135°, 315° 15. $\cos^2 A$ 16. $x^2 + y^2 - 6x - 4y = 0$
17. $\{3, -\frac{3}{2} + \frac{1}{2}i\sqrt{3}, -\frac{3}{2} - \frac{1}{2}i\sqrt{3}\}$ 18. -3 19. 440 20. 320

Test 26. 1. $\{-\frac{5}{2} \leqq x \leqq 1\}$ 2. $x = \frac{3}{2}$ 3. $x = 4,\ y = -1$ 4. $x^2 - 4x + 7 = 0$ 5. Center $(-2, 0)$, radius 2 6. $(-\frac{3}{2}, 3)$; line is tangent to ellipse. 7. Hyperbola 8. $\tan \theta = \pm\frac{\sqrt{1 - m^2}}{m}$ 9. $6\sqrt{7}$ in. 10. a. $|x| \leqq 3$ b. $0 \leqq y \leqq 6$ 11. $\{210°, 33°, 270°\}$ 12. $\tan \theta$ 13. Between -2 and -1 14. 9 15. $\{-2 < x < 0\} \cup \{x > 2\}$ 16. $\{\frac{1}{2}, -\frac{1}{2} + \frac{1}{2}i\sqrt{7}, -\frac{1}{2} - \frac{1}{2}i\sqrt{7}\}$ 17. 294 18. 5 19. a. $4\frac{1}{2}$ b. $S_n = \frac{1}{2}(3^n - 1)$ 20. $x^2 + y^2 - 16x + 48 = 0$, or $(x - 8)^2 + y^2 = 16$; locus is circle with center $(8, 0)$ and radius 4.

Test 27. 1. a. $-\frac{\sqrt{3}}{2}$ b. 1 c. -1 2. $\left\{\frac{7\pi}{6}, \frac{11\pi}{6}\right\}$ 3. $\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$ 4. a. 0 b. $-\frac{2\pi}{3}$ 5. a. $\frac{4}{3}$ rad. b. 96 sq.in. c. 76°24' 6. $\{0 \leqq \theta < \frac{\pi}{4}\} \cup \{\frac{5\pi}{4} < \theta \leqq 2\pi\}$ 7. a. 0 b. $\frac{1}{2}$ 8. $\frac{1}{5}$ 9. $\tan A$ 10. 18°26' 11. a. 1 b. $\sec \frac{1}{2}x$ 12. $\{\frac{\pi}{3}, \frac{4\pi}{3}\}$

Test 28. 1. a. $\frac{1}{2}\sqrt{2}$ b. $\frac{1}{2}\sqrt{3}$ c. $\frac{1}{2}$ 2. a. $\left\{\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}\right\}$ b. $\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\}$ 3. a. $\sin 2\theta$ b. $\cos A$ c. 2 4. a. Domain is $\{x \mid -1 \leqq x \leqq 1\}$; range is $\left\{y \mid -\frac{\pi}{2} \leqq y \leqq \frac{\pi}{2}\right\}$ b. $+\frac{2\sqrt{2}}{3}$ 5. a. $\frac{1}{2}$ b. $\frac{4}{5}$

Test 29. 1. $\{3, 1\}$ 2. $y = \frac{ab}{a^2 + 1}$ 3. $\{x \geqq -2\}$ 4. $(x + 2y + 3)(x + 2y - 3)$ 5. Domain is $\{x \mid |x| \leqq 2\}$ 6. $3\frac{1}{2}$ 7. $\frac{1}{9}$ 8. $x + y + 1 = 0$ 9. $\Delta y = 4$ 10. 11 11. 2.6 12. $\frac{1}{6}\sqrt{3}$ 13. -21 14. $x + 2y - 1 = 0$ 15. $f(x) = 6 - 3x$ 16. $\log (\frac{x^3}{y}) = 3a - b$ 17. $(\frac{5}{3}, \frac{25}{3})$ 18. $x^2 + y^2 + 2x - 4y = 0$ 19. $D = 0$ 20. $2\sqrt{21}$ in. 21. $(2x - 1)(x + 1)^2$; $x < \frac{1}{2}$ but $x \neq -1$. 22. $(3, -2)$ 23. $S_n = n^2 + 2n$ 24. 21 25. $\frac{2}{3}$ 26. 2 27. a. 1 b. $-\frac{\sqrt{3}}{2}$ 28. $\sin x$ 29. $\sin 2A$ 30. $\frac{1}{2}$

Test 30. 1. $x \leqq -5$ 2. $(3x + 2)(2x + 1)(2x - 1)$ 3. $\left\{0, \frac{b}{a - b}\right\}$ 4. $\{1.99 < x < 2.01\}$ 5. $2 - \sqrt{3}$ 6. $\frac{a}{b}$

7. $\frac{1}{16}$ 8. 2^n 9. $2\sqrt{5}$ units 10. $0 \leqq y \leqq 3$ 11. $2x + y - 6 = 0$

12. 5 13. 2 14. 15 15. 1000 16. -10

17. $\{x \leqq -1\} \cup \{x \geqq 4\}$ 18. Center $(\frac{3}{2}, -2)$, radius $\frac{5}{2}$. 19. $7\frac{1}{5}$

20. $6\sqrt{2}$ in. 21. $f(x) = 2x^2 - 8x + 6$ 22. 17 23. 2 and 3

24. 230 25. 9 26. $\pm\frac{1}{4}\sqrt{7}$ 27. a. $-\frac{\sqrt{3}}{2}$ b. -1 28. $\{\frac{3\pi}{4}, \frac{7\pi}{4}\}$

29. 1 30. sec 2θ

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